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Energy amplification in channel flows with stochastic excitation

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We investigate energy amplification in parallel channel flows, where background noise is modeled as stochastic excitation of the linearized Navier–Stokes equations. We show analytically that the energy of three-dimensional streamwise-constant disturbances achieves $O(R^3)$ amplification. Our basic technical tools are explicit analytical calculations of the traces of solutions of operator Lyapunov equations, which yield the covariance operators of the forced random velocity fields. The dependence of these quantities on both the Reynolds number and the spanwise wave number are explicitly computed. We show how the amplification mechanism is due to a coupling between wall-normal velocity and vorticity disturbances, which in turn is due to nonzero mean shear and disturbance spanwise variation. This mechanism is viewed as a consequence of the non-normality of the dynamical operator, and not necessarily due to the existence of near resonances or modes with algebraic growth. [DOI: 10.1063/1.1398044]

I. INTRODUCTION

In the past several years, there has been an intensive investigation of disturbance energy growth in subcritical channel and boundary layer flows. It has been observed that otherwise linearly stable flows can exhibit very large three-dimensional (3-D) disturbance energy growth. This has been proposed as a possible mechanism for ''bypass'' or natural transition to turbulence in shear flows.[1–4] This mechanism is primarily due to linear amplification of disturbances energized by the background mean shear. It occurs in the absence of nonlinear effects, and bypasses the primary/secondary instabilities scenarios.[5–7]

One approach to this problem is to consider the energy growth of ''worst case'' initial flow disturbances. This is the point of view adopted in Refs. 2–4, where it is shown that transient energy growth can achieve maxima of $O(R^2)$ for certain favorably configured initial states. These maxima occur at times which are $O(R)$. A transition scenario can then be proposed where such large transient growth causes an exit from the basin of attraction of the linearly stable laminar flow. It has been found that such a scenario requires disturbances with energies two orders of magnitude lower than those of Tollmein–Schlichting waves.[8] Though large energy growth is demonstrated, one is left with the question of how does nature conspire to set up such worst case initial conditions.

A second approach partially answers this difficult question. In this approach, one considers the excitation of the linearized Navier–Stokes equations by a stochastic random field which enters as a forcing term.[9,10] This random excitation can model background noise which exists in naturally transitioning flows. A Karhunen–Loeve analysis of the resulting second-order statistics brings out dominant structures, which have the structure of streamwise vortices and streaks.[9] In this latter work, it was also observed that variance (energy) growth for streamwise constant vortices is $O(R^3)$, while it is $O(R^{3/2})$ for disturbances with streamwise variations. These observations were made through numerical approximations of the underlying PDEs, and solutions of corresponding Lyapunov equations for the covariance matrices.

We point out that in this stochastic excitation model which we consider in this paper, it is more appropriate to speak of *energy amplification* rather than energy growth. The dynamical equations can be thought of as representing a system where background noise is regarded as an ''input,'' and the resulting forced random velocity field as the ''output.'' The ratio of the output energy (variance) to that of the input is defined as the energy amplification of the system. Systems with very large amplification (as is the case with high shear flow) are very sensitive to noise inputs. Such noise will then determine the dominant structure of the observed output under ''naturally noisy'' conditions. Numerical experiments in channel flow[11] indicate that even small amounts of background noise due to round-off error can alter the modes present in transition. We also mention that versions of this input–output point of view were essentially adopted in some of the recent turbulence control for drag reduction studies.[12–14]

In this paper we will analyze stochastically excited channel flow and analytically demonstrate the energy amplification can be $O(R^3)$ for 3-D disturbances. This is done for three-dimensional streamwise-constant disturbances only (the so-called two-dimensional, three component model[15]). This particular choice is motivated by the observations through numerical approximations[9] that streamwise constant disturbances have the most energetic growth.

By investigating properties of the solutions of the underlying operator Lyapunov equations we obtain additional insight into the energy amplification process. We show that $O(R^3)$ amplification is an inherent property of streamwise constant disturbances in any general three-dimensional par-

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allel channel flow. This amplification property is independent of the particular mean flow, and is a consequence of a coupling between the perturbed wall-normal velocity and vorticity. This coupling is proportional to the background laminar flow shear and the spanwise variation of the disturbance. This growth is also independent of whether there are near resonances or modes with algebraic growth. The last observation has also been made in Ref. 1 by analogy with a two-dimensional non-normal model, but we show it here for the full linearized PDE of channel flow disturbances.

For the specific case of Couette flow we carry out a detailed analysis of the dependence of variance amplification on the spanwise wave number. The peak of this "frequency response" represents flow structures that are dominant in a flow field excited by a broad-band, stochastic forcing field. This peak corresponds to flow structures that are streamwise vortices and streaks.

The calculation of these energy frequency response is performed by solving certain infinite dimensional Lyapunov equations, and computing the trace of the resulting operators. We show how these traces can be calculated analytically. This is made possible by a careful analysis of the underlying two point boundary value problems arising from the linearized Navier–Stokes equations.

To summarize our results, we obtain an explicit expression for energy amplification in stochastically excited linearized channel flows in the following form:

$$E = f_1(k_z)R + f_2(k_z)R^3,$$

where k_z is the spanwise wave number and R is the Reynolds number. We obtain the function f_1 explicitly, and the function f_2 in terms of a rapidly convergent series. This expression is valid for disturbances with streamwise wave number $k_x = 0$, and reflects th disturbance energy averaged both temporally and in the wall-normal direction.

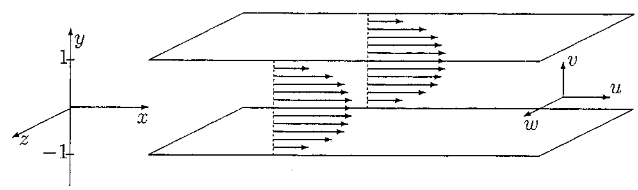

FIG. 1. Three-dimensional channel flow.

Our presentation is organized as follows: We first summarize the analysis of stochastically excited linearized Navier–Stokes equations. We illustrate how the random flow field covariance operator can be obtained from the solution of an operator Lyapunov equation. We then analyze the block decomposition of the Lyapunov equation, and show by a scaling argument that streamwise constant disturbances have $O(R^3)$ energy amplification for all spanwise wave numbers. We then devote the remainder of the paper to the analytic evaluation of the dependence on the spanwise wave number, which involves computing the traces of Lyapunov equation solutions. We close by summarizing our conclusions, commenting on the input–output view of transition in shear flows.

II. LINEARIZED NAVIER–STOKES EQUATIONS AND ENERGY AMPLIFICATION

We begin by considering the nondimensionalized linearized incompressible Navier–Stokes equations which describe the dynamics of flow field *perturbations* near a laminar (or mean) flow profile $U_m(x,y,z) = U(y)$, $-1 \leqslant y \leqslant 1$ (flow between two parallel infinite plates, see Fig. 1 for the geometry). After eliminating the pressure field and rewriting the equations in terms of wall-normal velocity v and wall-normal vorticity $\omega := \partial u/\partial z - \partial w/\partial x$ perturbations, we obtain[16]

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \left(-\Delta^{-1}U\frac{\partial}{\partial x}\Delta + \Delta^{-1}U''\frac{\partial}{\partial x} + \frac{1}{R}\Delta^{-1}\Delta^2\right) & 0 \\ \left(-U'\frac{\partial}{\partial z}\right) & \left(-U\frac{\partial}{\partial x} + \frac{1}{R}\Delta\right) \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} =: \begin{bmatrix} \mathcal{L} & 0 \\ \mathcal{C} & \mathcal{S} \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix} =: \mathcal{A}\begin{bmatrix} v \\ \omega \end{bmatrix}, \tag{1}$$

where $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian, U' is the derivative of U with respect to y, and R is the Reynolds number based on the maximum velocity of the laminar flow profile. (The notation "=:" means that the right-hand side is defined as the left-hand side. Therefore Eq. (1) also serves to define the operators $\mathcal{L}$, $\mathcal{S}$, and $\mathcal{C}$.) By the notation $T_1^{-1}T_2$, where T_1 and T_2 are PDE operators, we mean the operator $T_1^{-1}T_2 : g \mapsto f$, where f is the solution the inhomogeneous differential equation $T_1 f = T_2 g$. Such an operator is well defined if and only if there exists a unique solution to the differential equation for every g in the domain of the operator.

The velocity and vorticity perturbation fields are initially allowed to vary temporally and in all three spatial dimensions, and are thus functions of (x,y,z,t). In a later section, we will make some restrictions on the allowable perturbations. The boundary conditions on these fields are

$$\begin{aligned} v(x,\pm 1,z,t) &= \frac{\partial}{\partial y}(x,\pm 1,z,t) = 0, \\ \omega(x,\pm 1,z,t) &= 0, \quad \forall x,z,t \in \mathbb{R}. \end{aligned} \tag{2}$$

$\mathcal{L}$ and $\mathcal{S}$ are termed the Orr–Sommerfeld and Squire operators, respectively. $\mathcal{C}$ is an operator that represents the coupling from wall-normal velocity to wall-normal vorticity and, as we show later, is responsible for the $O(R^3)$ amplification of disturbance energy.

This particular representation of the equations is most useful since these two fields are not subject to any additional constraints other than the boundary conditions. In system theoretic language, the true state space of the linearized Navier–Stokes equations is the space of all unconstrained wall-normal velocity and vorticity fields. The above-mentioned form of the equations is sometimes referred to as the evolution form.

An examination of the above-mentioned model indicates that the generator $\mathcal{A}$ is invariant to translations in the x and z directions. The appropriate analysis is then performed using a Fourier transformation in those variables which yields

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}=\begin{bmatrix}\left(-ik_x\Delta^{-1}U\Delta+ik_x\Delta^{-1}U''+\frac{1}{R}\Delta^{-1}\Delta^2\right) & 0\\ (-ik_zU') & \left(-ik_xU+\frac{1}{R}\Delta\right)\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}, \tag{3}$$

where k_x,k_z are the spatial frequencies (wave numbers) in the x,z directions, respectively, and $\hat{v},\hat{\omega}$ are the transformed wall-normal velocity and vorticity fields, e.g.,

$$\hat{v}(k_x,y,k_zt):=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v(x,y,z,t)e^{-i(xk_x+zk_z)}dx\,dz.$$

With a slight abuse of notation, we will refer to the transformed generator in (3) by $\mathcal{A}$, the same symbol used for the generator in (1). We also use the same symbols for all the constituent operators, e.g., the Laplacian is now $\Delta=(\partial^2/\partial y^2-k_x^2-k_z^2)$. Note that the above-mentioned transformed evolution equation is now a one-dimensional PDE parametrized by the two parameters k_x,k_z.

Remark: In the literature, evolution equation (3), together with the assumption that temporal growth is exponential, leads to the so-called normal modes, and stability is considered equivalent to the nonexistence of growing normal modes. we note here and that the above-mentioned Fourier transformation is also appropriate for the more general energy growth/amplification analysis we carry out in this paper. It can be shown[17] that the original 3-D system (1) is exponentially stable if and only if the transformed system (3) (which is a one-dimensional PDE) is exponentially stable for all k_x,k_z (with an additional technical condition). However, in this paper we are interested in energy amplification properties rather than modal stability. It turns out that since the Fourier transformation preserves quadratic forms, the transformed model (3) captures all the energy growth and amplification properties of the original model (1). We refer the reader to Ref. 17 for the details.

For any given k_x,k_z, we can define the *kinetic energy density* of a harmonic perturbation at a given time by

$$E:=\frac{k_xk_z}{16\pi^2}\int_{-1}^{1}\int_{0}^{2\pi/k_x}\int_{0}^{2\pi/k_z}(u^2+v^2+w^2)dz\,dx\,dy,$$

which is a mean square integral averaged over a ''box'' of one wavelength side. Using an integration by parts and the definitions of the Fourier transform, this expression can be evaluated in terms of $\hat{v}$ and $\hat{w}$ by

$$\begin{aligned}E&=\frac{1}{8}\int_{-1}^{1}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}^*\begin{bmatrix}I-\frac{1}{k_x^2+k_z^2}\frac{\partial^2}{\partial y^2} & 0\\ 0 & \frac{1}{k_x^2k_z^2}I\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}dy\\ &=:\langle\psi,\mathcal{Q}\psi\rangle,\end{aligned} \tag{4}$$

where we renamed the state as ψ, and the energy form is given by the linear operator $\mathcal{Q}$, i.e.,

$$\psi:=\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix},\qquad \mathcal{Q}:=\frac{1}{8(k_x^2+k_z^2)}\begin{bmatrix}-\Delta & 0\\ 0 & I\end{bmatrix}.$$

This energy form defines an inner product on the state space such that the kinetic energy density is a norm on this space. Due to its importance, we denote this inner product by

$$\langle\psi_1,\psi_2\rangle_e:=\langle\psi_1,\mathcal{Q}\psi_2\rangle,$$

where the inner product on the right-hand side is the standard $L^2[-1,1]$ inner product given by (4). We also recall the important concept of the adjoint of an operator with respect to a given inner product. Given an operator $\mathcal{H}$ on a Hilbert space with an inner product $\langle\cdots\rangle_e$, its adjoint operator, denoted by $\mathcal{H}^*$is defined by the relation

$$\langle\psi_1,\mathcal{H}\psi_2\rangle_e=\langle\mathcal{H}^*\psi_1,\psi_2\rangle_e,$$

which must hold for all ψ_1,ψ_2 in the Hilbert space. (A Hilbert space is a linear function space with an inner product and a completeness property, this latter property does not play an important role in this paper and will not be discussed further.) The adjoint is a generalization of the complex conjugate transpose of a matrix and has similar properties. Using the adjoint, one defines the *singular values* of an operator $\mathcal{H}$ as the square root of the eigenvalues of $\mathcal{H}\mathcal{H}^*$ (or equivalently $\mathcal{H}^*\mathcal{H}$). Similarly the left (respectively, right) singular vectors of $\mathcal{H}$ are defined as the eigenvectors of $\mathcal{H}\mathcal{H}^*$ (respectively, $\mathcal{H}^*\mathcal{H}$). When an operator is normal (i.e., when $\mathcal{H}\mathcal{H}^*=\mathcal{H}^*\mathcal{H}$, or equivalently, when its eigenvectors are orthogonal), its singular values and eigenvalues are equal (up to change of sign). When an operator is non-normal, the singular and eigenvalues can be quite different, and in general bear no relationship to each other. In fact, much of the energy growth properties of shear flows that have been studied in the past decade are due to the fact that the eigenvalues of the propagator $e^{t\mathcal{A}}$ are quite different from the eigenvalues of $e^{t\mathcal{A}}e^{t\mathcal{A}^*}$. The former is the subject of classical linear hydrodynamic stability theory using modes, and the latter is the appropriate measure of energy growth.

We will not recap the energy growth results in the literature, referring the reader instead to some of the original work,[2,4] and a review article.[1] In that work, the model (3) is considered, and favorably configured initial conditions are found that yield the maximum energy growth in a given time period. We will instead consider the same model, but with external forcing and zero initial conditions, and analyze the effects of persistent random forcing disturbances on the flow field. This was originally done in Ref. 9 with a Galerkin approximation of the PDE. We will consider the same model and derive analytical expressions for the energy amplification. We will also clarify the amplification mechanism in terms of some simple properties of the underlying operators.

We will consider the transformed linearized Navier–Stokes equations (3), with an additional forcing term on the right-hand side

$$\frac{\partial}{\partial t}\psi=\mathcal{A}\psi+d,\quad d:=\begin{bmatrix} d_v \\ d_\omega \end{bmatrix},$$

where d is a forcing term with two components, the wall-normal velocity d_v and vorticity d_ω forcing terms, respectively. These terms account for external body forces imposed on the flow such as free stream disturbances. We will assume d to be a spatiotemporal random process (i.e., a random field) with a unit covariance as measured by the energy form. This setup reflects the simplest possible model of external stochastic excitation. More refined models may include more specifications on the statistics of d. For example, a simple model of distributed wall roughness can be obtained by assuming d to be a random field whose intensity peaks near the walls. This would model the effect of distributed random surface texture on the flow field near the wall.

With such forcing, the velocity field becomes a random field whose statistics are determined by the dynamics of the system. It is a standard fact from the theory of linear systems forced with second-order stochastic processes that the stationary (i.e., steady state) covariance operator of ψ is given by

$$V=\int_0^\infty e^{t\mathcal{A}}e^{t\mathcal{A}^*}\,dt.$$

In order to evaluate this covariance operator, it is not necessary to perform the above-mentioned integration. It can be shown that it is the unique solution of the following operator Lyapunov equation:

$$\mathcal{A}V+V\mathcal{A}^*=-I. \tag{5}$$

For reference, the above facts from the theory of linear systems driven by stochastic noise are elaborated on in Appendix A.

Note that the covariance operator V in (5) is a function of the wave numbers k_x,k_z. More precisely, let the disturbance d be given by

$$d(x,y,z,t)=\overline{d}(y,t)\Re(e^{i(xk_x+zk_z)}),$$

where $\{\overline{d}(y,t)\}$ is a white, unit variance, temporally stationary, second-order random field. Such disturbances are stochastic in the wall-normal and temporal directions and pure harmonic in the streamwise and spanwise direction, but we will simply refer to them as harmonic disturbances. The resulting random velocity field will also be temporally stationary and pure harmonic in the x,z directions, and its variance is given by trace $[V(k_x,k_x)]$. Thus trace $[V(k_x,k_z)]$ provides the variance (energy) amplification of harmonic disturbances. This quantity is referred to as the *ensemble average energy density* of a harmonic disturbance at a given k_x and k_z by Ref. 9.

III. DEPENDENCE OF VARIANCE AMPLIFICATION ON THE REYNOLDS NUMBER

We now consider a general channel flow (described by its corresponding laminar flow profile U), and show that for 3-D streamwise constant perturbations, energy amplification is $O(R^3)$. The dynamics of streamwise constant perturbations ar given by Eq. (3) at $k_x=0$, which becomes

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}=\begin{bmatrix}\left(\frac{1}{R}\Delta^{-1}\Delta^2\right) & 0\\ (-ik_zU') & \left(\frac{1}{R}\Delta\right)\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix},$$

$$\Delta=\left(\frac{\partial^2}{\partial y^2}-k_z^2I\right). \tag{6}$$

For different channel flow, the corresponding models differ only in the U' term. We will now demonstrate that $O(R^3)$ energy amplification is achieved for any channel flow with nonzero shear $(U'\neq 0)$, and a disturbance with nonzero spanwise variation $(k_z\neq 0)$.

We view the generator in (6) as a ''lower triangular'' 2×2 block operator, and we investigate the solution of the Lyapunov equation (5) ''block by block.'' Specifically, we re-write (5) as

$$\begin{bmatrix}\mathcal{A}_{11} & 0\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}\begin{bmatrix}V_{11} & V_0^*\\ V_0 & V_{22}\end{bmatrix}+\begin{bmatrix}V_{11} & V_0^*\\ V_0 & V_{22}\end{bmatrix}\begin{bmatrix}\mathcal{A}_{11}^* & \mathcal{A}_{21}^*\\ 0 & \mathcal{A}_{22}^*\end{bmatrix}=\begin{bmatrix}-I & 0\\ 0 & -I\end{bmatrix}.$$

This equation can equivalently be written as the following set of coupled equations:

$$\mathcal{A}_{11}V_{11}+V_{11}\mathcal{A}_{11}^*=-I, \tag{7}$$

$$\mathcal{A}_{22}V_0+V_0\mathcal{A}_{11}^*=-\mathcal{A}_{21}V_{11}, \tag{8}$$

$$\mathcal{A}_{22}V_{22}+V_{22}\mathcal{A}_{22}^*=-(I+\mathcal{A}_{21}V_0^*+V_0\mathcal{A}_{21}^*). \tag{9}$$

The block lower triangular structure of the generator implies that the above-mentioned equations are coupled in a convenient manner; Eq. (7) is a Lyapunov equation which can be solved for V_{11}, then the Sylvester equation (8) can be solved for V_0, and finally the results can be used to solve the Lyapunov equation (9) for V_{22}.

In the next section we will illustrate methods to explicitly solve a subset of the above-mentioned equations, and compute the trace of the solution of the remaining equation. These commutations will be done for the case of Couette flow. However, the $O(R^3)$ amplification can be seen as a general property of nonzero shear channel flows from the following general argument. Let us denote the generator blocks as follows:

$$\begin{bmatrix} \frac{1}{R}\mathcal{L} & 0 \\ \mathcal{C} & \frac{1}{R}\mathcal{S} \end{bmatrix} := \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where we have denoted the R-indepenent, normalized Orr–Sommerfeld and Squire operators by $\mathcal{L}$ and $\mathcal{S}$, respectively. In this notation, the coupled equations (7)–(9) become

$$\mathcal{L}V_{11} + V_{11}\mathcal{L}^* = -IR, \tag{10}$$

$$\mathcal{S}V_0 + V_0\mathcal{L}^* = -\mathcal{C}V_{11}R, \tag{11}$$

$$\mathcal{S}V_{22} + V_{22}\mathcal{S}^* = -(I + \mathcal{C}V_0^* + V_0\mathcal{C}^*)R. \tag{12}$$

Let us denote the solutions of Eqs. (10)–(12) at $R=1$ by P_{11}, P_0, and P_{22}, respectively. To analyze the last equation, we express $P_{22} = P_{22_1} + P_{22_2}$, and we write the equations for the R-independent P operators

$$\mathcal{L}P_{11} + P_{11}\mathcal{L}^* = -I, \tag{13}$$

$$\mathcal{S}P_0 + P_0\mathcal{L}^* = -\mathcal{C}P_{11}, \tag{14}$$

$$\mathcal{S}P_{22_2} + P_{22_2}\mathcal{S}^* = -(\mathcal{C}P_0^* + P_0\mathcal{C}^*), \tag{15}$$

$$\mathcal{S}P_{22_1} + P_{22_1}\mathcal{S}^* = -I. \tag{16}$$

Due to the linearity of the equations, we see that (10) implies that $V_{11} = P_{11}R$. Substituting this in (11), we observe that $V_0 = P_0R^2$, and finally (12) implies that $V_{22_1} = P_{22_1}R$, while $V_{22_2} = P_{22_2}R^3$.

We can now write the energy amplification as

$$\mathrm{tr}(V) = \mathrm{tr}(V_{11}) + \mathrm{tr}(V_{22}) = [\mathrm{tr}(P_{11}) + \mathrm{tr}(P_{22_1})]R + \mathrm{tr}(P_{22_2})R^3. \tag{17}$$

The $O(R^3)$ energy amplification now appears due to the operator P_{22_2}. We note that this operator is zero whenever $\mathcal{C} := ik_zU'$ is zero, which happens either in the absence of shear (i.e., $U'=0$), or when there is no spanwise variation in the perturbation (i.e., $k_z=0$).

Formula (17) illustrates nicely the dependence of energy amplification on both R and k_z. The dependence on k_z is expressed through the traces of the P operators, which are independent of R. We summarize the above-mentioned development in the following statement.

Theorem 1: *For any parallel channel flow, the energy amplification of streamwise constant disturbances is given by*

$$\begin{aligned}\mathrm{tr}(V(k_z)) &= [\mathrm{tr}(P_{11}(k_z)) + \mathrm{tr}(P_{22_1}(k_z))]R \\ &\quad + \mathrm{tr}(P_{22_2}(k_z))R^3 \\ &=: f_1(k_z)R + f_2(k_z)R^3.\end{aligned} \tag{18}$$

The dependence on k_z is captured by the two functions f_1, f_2. It is easy to see from (18), (16), and (13) that f_1 depends only on $\mathcal{L}$ and $\mathcal{S}$, and thus is the same for all channel flows. Only f_2 depends on the form of the laminar flow (or mean shear) through its dependence on U'. We note that for high Reynolds numbers, the dependence on k_z is essentially dominated by the f_2 term as the contribution of the linear (in R) term becomes negligible.

The $O(R^3)$ term's coefficient is trace(P_{22_2}), which comes from the solutions of (13)–(15). From these equations it can be shown that trace(P_{22_2}) is nonzero whenever both U' and k_z are nonzero. Note that no assumptions on the existence of any resonances between the eigenvalues of $\mathcal{S}$ and $\mathcal{L}$ were invoked. This illustrates that $O(R^3)$ energy amplification can occur even in the absence of near-resonant modes or so-called algebraic growth.[16]

The previous development is applicable to any channel flow problem. We have thus established that $O(R^3)$ amplification is inherent in 3-D streamwise constant perturbations. In Sec. IV we proceed to compute analytically the function $f_1(k_z)$ (which is the same for all channel flows) and the function $f_2(k_z)$ for Couette flow.

IV. TRACE COMPUTATIONS

Although in general it is quite difficult to solve operator Lyapunov equations without resorting to finite dimensional approximations of the operators, we will show in this section that *traces* of the solutions can be computed exactly. We note here that there are two methods by which these traces can be computed. The first involves the spectral decompositions of the Orr–Sommerfeld and Squire ($\mathcal{L}$ and $\mathcal{S}$) operators, both of which are well known.[18] By writing the operator's matrix representations using the eigenfunctions of $\mathcal{L}$ and $\mathcal{S}$ as bases for v and ψ, respectively, both $\mathcal{L}$ and $\mathcal{S}$ will have diagonal representations. Equations (13)–(16) can then in principle be solved for the matrix representations of the P operators. This method is tractable when the operators involved have simple spectral decompositions. The present difficulty is that the eigenvalues of the operator $\mathcal{L}$ are not known explicitly, but only as zeros of certain transcendental functions.[18]

The second method and the one which we develop in this paper is based on certain properties of the operator trace, and the use of so-called state-space realizations commonly used in control theory.[19] These techniques circumvent the detailed spectral analysis of the operator $\mathcal{L}$. In Sec. IV A, we investigate the properties of some special Lyapunov equations, and show that Eqs. (13) and (16) are easily solvable for P_{11} and P_{22_1}. While we will not be able to solve for P_{22_2} explicitly, we will be able to express its trace as an easily computable series. In Sec. IV B we turn to the explicit evaluation of the traces of P_{11} and P_{22_1}. We show how they can be computed without knowledge of the corresponding spectra by considering ''state space'' representations of the underlying two point boundary value problems. In Sec. IV C, the series expressing the trace of P_{22_2} will be evaluated by again considering the underlying two point boundary value problems.

Before we begin, we will define the $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{C}$ operators more precisely. The Squire operator $\mathcal{S}$ has homogeneous Dirichlet boundary conditions and domain $\mathcal{D}(\mathcal{S})$ given by

$$\mathcal{S}:=\left(\frac{\partial^2}{\partial y^2}-k_z^2\right),$$

$$\mathcal{D}(\mathcal{S}):=\{g\in L^2[-1,1];g^{(2)}\in L^2[-1,1],g(\pm 1)=0\}.$$

It is well known that it is self-adjoint, and has the following set of orthonormal eigenfunctions $\{\phi_n\}$ with corresponding eigenvalues $\{\gamma_n\}$,

$$\phi_n(y):=\sin\left(\frac{n\pi}{2}(y+1)\right),\quad \gamma_n=-\left(\frac{n^2\pi^2}{4}+k_z^2\right),\quad n\geqslant 1. \tag{19}$$

The description of the Orr–Sommerfeld operator is more delicate since its eigenfunctions actually depend on k_z. We first begin by carefully defining the operator following Ref. 4. The underlying space $\mathbb{H}_{\mathrm{OS}}$ is the space of function g with second derivative in $L^2[-1,1]$, and $g(\pm 1)=0$. The inner product in $\mathbb{H}_{\mathrm{OS}}$ is given by

$$\langle g_1,g_2\rangle_{\mathrm{OS}}:=\langle g_1',g_2'\rangle_2+k_z^2\langle g_1,g_2\rangle_2=\langle g_1,(-\Delta)g_2\rangle_2,$$

where for emphasis we have denoted the standard inner product in L^2 by $\langle\cdots\rangle_2$. Note that this inner product gives precisely the contribution of the wall-normal velocity field to the energy density. The energy form in (4) can now be written

$$\langle\psi_1,\psi_2\rangle_e=\left\langle\begin{bmatrix}\hat{v}_1\\ \hat{\omega}_1\end{bmatrix},\begin{bmatrix}\hat{v}_1\\ \hat{\omega}_1\end{bmatrix}\right\rangle_e=\langle\hat{v}_1,\hat{v}_2\rangle_{\mathrm{OS}}+\langle\hat{\omega}_1,\hat{\omega}_2\rangle_2.$$

The domain of $\mathcal{L}$ is inside of $\mathbb{H}_{\mathrm{OS}}$, namely

$$\mathcal{D}(\mathcal{L}):=\{g\in\mathbb{H}_{\mathrm{OS}};\ \ g'(\pm 1)=0,\ \ g^{(4)}\in L^2[-1,1]\},$$

and $\mathcal{L}$ is defined a the operator that maps $g\mapsto f$ for $f,g\in\mathcal{D}(\mathcal{L})$ by

$$\left(\frac{\partial^2}{\partial y^2}-k_z^2\right)f=\left(\frac{\partial^4}{\partial y^4}-2k_z^2\frac{\partial^2}{\partial y^2}+k_z^4\right)g.$$

It is not difficult to verify that $\mathcal{L}$ is self-adjoint with a discrete spectrum, and is negative definite (with respect to the inner product of $\mathbb{H}_{\mathrm{OS}}$), and thus has only negative real eigenvalues and generates a stable evolution. The self-adjointness of $\mathcal{L}$ implies that it can be diagonalized by using its eigenfunctions as an orthonormal basis of $\mathbb{H}_{\mathrm{OS}}$. This eigenvalue problem was studied in Ref. 18, to which we refer the interested reader for details. In this paper we circumvent the need for the detailed spectral analysis of $\mathcal{L}$. We will only need to know that it is self-adjoint, trace class (an operator is trace class if it has finite trace), and has a discrete spectrum.

The coupling operator $\mathcal{C}$: $\mathbb{H}_{\mathrm{OS}}\rightarrow L^2[-1,1]$ is given by $\omega=\mathcal{C}v$,

$$\omega(y)=-ik_zU'(y)v(y),$$

and is a bounded operator as is evidenced by the following bound:

$$\|\omega\|_2\leqslant\|v\|_{\mathrm{OS}}\left(\max_{y\in[-1,1]}\|U'(y)\|\right),$$

which can be easily derived from the definitions. In the sequel, we will need the adjoint of $\mathcal{C}$ which we now evaluate. The adjoint $\mathcal{C}^*:L^2[-1,1]\rightarrow\mathbb{H}_{\mathrm{OS}}$ satisfies

$$\langle\omega,\mathcal{C}v\rangle_2=\langle\mathcal{C}^*\omega,v\rangle_{\mathrm{OS}}=-\langle\mathcal{C}^*\omega,\Delta v\rangle_2=-\langle\Delta\mathcal{C}^*\omega,v\rangle_2,$$

which implies that it must be given by

$$v=\mathcal{C}^*\omega\Leftrightarrow v(y)=-ik_z\Delta^{-1}U'(y)w(y).$$

A. Special Lyapunov equations

Our analysis is based on the following properties of some special Lyapunov equations.

Lemma 2: Let A be a possibly unbounded operator that generates an exponentially stable semigroup $\{e^{tA}\rangle$, *then*

(1) *The unique solution of the operator Lyapunov equation* $AP+PA=-I$, is $P=-\frac{1}{2}A^{-1}$.

(2) *Given a self-adjoint operator Q, such that both Q and* $\{e^{tA}\}$ *are trace class, then the unique solution of the operator Lyapunov equation* $AP+PA=-Q$ *satisfies*

$$\mathrm{trace}(P)=-\tfrac{1}{2}\mathrm{trace}(A^{-1}Q).$$

We apply the first part of this lemma to Eqs. (13) and (16). As is well known (see the following), both $\mathcal{L}$ and $\mathcal{S}$ ar self adjoint operators that generate trace class and exponentially stable semigroups, thus $\mathcal{L}^*=\mathcal{L}$, $\mathcal{S}^*=\mathcal{S}$, and both equations [(13) and (16)] can then be solved to yield

$$P_{11}=-\tfrac{1}{2}\mathcal{L}^{-1},\quad P_{22_1}=-\tfrac{1}{2}\mathcal{S}^{-1}.$$

On the other hand, solving (15) is significantly more delicate, and requires solving (14) first, which we write formally as

$$P_0=\int_0^\infty e^{t\mathcal{S}}\mathcal{C}P_{11}\,e^{t\mathcal{L}}\,dt=-\frac{1}{2}\int_0^\infty e^{t\mathcal{S}}\,\mathcal{C}e^{t\mathcal{L}}\,dt\,\mathcal{L}^{-1},$$

since $\mathcal{L}^{-1}$ commutes with $e^{t\mathcal{L}}$. It does not seem possible to explicitly solve (15), but using the second part of the above-given lemma, we can write

$$\mathrm{trace}(P_{22_2})=-\tfrac{1}{2}\mathrm{trace}(\mathcal{S}^{-1}(\mathcal{C}P_0^*+P_0\mathcal{C}^*)) \tag{20}$$

$$=\frac{1}{4}\mathrm{trace}\left(\mathcal{S}^{-1}\left(\mathcal{C}\mathcal{L}^{-1}\int_0^\infty e^{t\mathcal{L}}\mathcal{C}^*e^{t\mathcal{S}}\,dt+\int_0^\infty e^{t\mathcal{S}}\mathcal{C}e^{t\mathcal{L}}\,dt\,\mathcal{L}^{-1}\mathcal{C}^*\right)\right) \tag{21}$$

$$=\frac{1}{2}\mathrm{trace}\left(\mathcal{S}^{-1}\mathcal{C}\mathcal{L}^{-1}\int_0^\infty e^{t\mathcal{L}}\mathcal{C}^*e^{t\mathcal{S}}\,dt\right), \tag{22}$$

where Eq. (22) is arrived at by using the commutativity of $\mathcal{L}^{-1}$ with $e^{t\mathcal{L}}$, and the fact that trace $(AB)=\mathrm{trace}(BA)$ for any two trace class operators A,B.

The inherent difficulty is that one cannot compute the integral in (22) explicitly. However as we now show, the trace can still be computed at the expense of doing a spectral decomposition of either $\mathcal{L}$ or $\mathcal{S}$. We of course choose the spectral decomposition of $\mathcal{S}$ since it is by far simpler.

Lemma 3: Let $\{\gamma_n\}$, $\{\phi_n\}$ *be the eigenvalues and eigenfunctions of the operator* $\mathcal{S}$*, then for any trace class operator* Q

$$\text{trace}\left(Q\int_0^\infty e^{t\mathcal{L}}\mathcal{C}^* e^{t\mathcal{S}}\,dt\right) = -\sum_{n=1}^{\infty} \langle \phi_n, Q(\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^*\phi_n\rangle.$$

Proof: Let the spectral decompositions of $\mathcal{S}$ and $\mathcal{L}$ be

$$\mathcal{S}=\sum_n \gamma_n E_n^{\mathcal{S}}, \quad \mathcal{L}=\sum_m \lambda_m E_m^{\mathcal{L}},$$

where $E_n^{\mathcal{S}}$ and $E_n^{\mathcal{L}}$ are the spectral projections of $\mathcal{S}$ and $\mathcal{L}$, respectively, e.g., $E_n^{\mathcal{S}} f := \langle \phi_n f\rangle \phi_n$. We can then compute

$$\begin{aligned}\int_0^\infty e^{t\mathcal{L}}\mathcal{C}^* e^{t\mathcal{S}}\,dt &= \int_0^\infty \left(\sum_m e^{\lambda_m t}E_m^{\mathcal{L}}\right)\mathcal{C}^*\left(\sum_n e^{\lambda_n t}E_n^{\mathcal{S}}\right)dt \\ &= \sum_n \sum_m \left(\int_0^\infty e^{(\lambda_m+\gamma_n)t}dt\right)E_m^{\mathcal{L}}\mathcal{C}^* E_n^{\mathcal{S}} \\ &= -\sum_n \sum_m \frac{1}{(\lambda_m+\gamma_n)} E_m^{L}\mathcal{C}^* E_n^{\mathcal{S}} \\ &= -\sum_n (\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^* E_n^{\mathcal{S}},\end{aligned}$$

where in the last equation we made a choice to recombine the spectral decomposition of $\mathcal{L}$. Now, the trace of any operator A can be calculated using any orthonormal basis set $\{\phi_i\}$ by $\text{trace}(A)=\Sigma_i\langle\phi_i, A\phi_i\rangle$. Therefore

$$\begin{aligned}\text{trace}\left(Q\int_0^\infty e^{t\mathcal{L}}\mathcal{C}^* e^{t\mathcal{S}}\,dt\right) &= -\text{trace}\left(Q\sum_n (\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^* E_n^{\mathcal{S}}\right) \\ &= -\sum_i \left\langle \phi_i, Q\sum_n (\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^* E_n^{\mathcal{S}}\phi_i\right\rangle \\ &= -\sum_i \langle \phi_i, Q(\mathcal{L}+\gamma_i I)^{-1}\mathcal{C}^*\phi_i\rangle.\end{aligned}$$

We now summarize the remaining required trace computations:

$$\text{trace}(P_{11}) = -\tfrac{1}{2}\text{trace}(\mathcal{L}^{-1}), \tag{23}$$

$$\text{trace}(P_{22_1}) = -\tfrac{1}{2}\text{trace}(\mathcal{S}^{-1}), \tag{24}$$

$$\text{trace}(P_{22_2}) = -\frac{1}{2}\sum_{n=1}^{\infty}\langle \phi_n, \mathcal{S}^{-1}\mathcal{C}\mathcal{L}^{-1}(\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^*\phi_n\rangle. \tag{25}$$

B. Computing the traces of $\mathcal{S}^{-1}$ and $\mathcal{L}^{-1}$

There is a particularly simple method to compute the trace of an operator given by a two point boundary value problem. This method is based on the use of certain state space realizations (first-order differential equation form), and works particularly well for the operators $\mathcal{S}^{-1}$ and $\mathcal{L}^{-1}$. This method is applicable to any operator specified by a two point boundary value problem,[20] and we only present here its application to the problem at hand.

First, we introduce the so-called state space realizations of the $\mathcal{S}^{-1}$ and $\mathcal{L}^{-1}$ operators. Recall the definition of the operators $\mathcal{S}$ and $\mathcal{L}$,

$$\mathcal{S}: g_1 \mapsto f_1 \Leftrightarrow f_1 = \left(\frac{\partial^2}{\partial y^2}-k_z^2\right)g_1, \quad g_1(\pm 1)=0,$$

$$\mathcal{L}: g_2 \mapsto f_2 \Leftrightarrow \left(\frac{\partial^2}{\partial y^2}-k_z^2\right)f_2 = \left(\frac{\partial^4}{\partial y^4}-2k_z^2\frac{\partial^2}{\partial y^2}+k_z^4\right)g_2,$$
$$g_2(\pm 1)=g_2'(\pm 1)=0,$$

their inverse are simply given by $\mathcal{S}^{-1}: f_1 \mapsto g_1$ and $\mathcal{L}^{-1}: f_2 \mapsto g_2$ as in the above-mentioned equations and boundary conditions. A set of state space realizations for the inverse can be given as follows:

$$\mathcal{S}^{-1}:\ z' = \begin{bmatrix} 0 & k_z^2 \\ 1 & 0\end{bmatrix} z + \begin{bmatrix}1\\0\end{bmatrix} f_1,$$

where

$$\begin{bmatrix} z_1 \\ z_2\end{bmatrix} := \begin{bmatrix} g_1' \\ g_1\end{bmatrix}, \quad z_2(\pm 1)=0,$$

$$g_1 = [0 \quad 1]z,$$

$$\mathcal{L}^{-1}:\ x' = \begin{bmatrix} 0 & 0 & 0 & -k_z^4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2k_z^2 \\ 0 & 0 & 1 & 0\end{bmatrix} x + \begin{bmatrix} -k_z^2 \\ 0 \\ 1 \\ 0\end{bmatrix} f_2, \tag{26}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} := \begin{bmatrix} g_2''' - 2k_z^2 g_2' - f_2' \\ g_2'' - 2k_z^2 g_2 - f_2 \\ g_2' \\ g_2\end{bmatrix},$$

$$g_2 = [0 \quad 0 \quad 0 \quad 1]\quad x, \quad \begin{bmatrix} x_3(\pm 1) \\ x_4(\pm 1)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Note that in both cases the realizations are such that the second half of the state variables is constrained to be zero at the boundary points $y=\pm 1$, while the first half is free. These realizations are in the so-called "observable canonical form."[19]

Now we apply the results of Ref. 20 on computing the trace given a state space realization to the above-mentioned systems. Note that since the corresponding realizations depend on k_z, the traces will also be functions of k_z. For both of these realizations, the formula in Ref. 20 was evaluated with the aid of the MAPLE/MATLAB symbolic computation package to yield the following:

$$\text{trace}(P_{22_1}) = -\frac{1}{2}\text{trace}(\mathcal{S}^{-1}) = \frac{2k_z \coth(2k_z) - 1}{k_z^2}, \quad (27)$$

$$\text{trace}(P_{11}) = -\tfrac{1}{2}\text{trace}(\mathcal{L}^{-1}) = -\frac{1}{2}\,\frac{4k_z^2 + \sinh(4k_z)k_z + (1-\cosh(4k_z))}{k_z^2(8k_z^2 + (1-\cosh(4k_z)))}. \quad (28)$$

In Fig. 2, we show plots of those two functions. It is interesting to note that both functions can be regarded as ratios of multinomials in the two variables k_z and e^{2k_z}. Note the characteristic dissipation spectrum for the trace of $\mathcal{S}^{-1}$. The spectrum of $\mathcal{L}^{-1}$ has a dissipation-like character for high wave numbers, but is somewhat different at low wave numbers. The difference between the two spectra can probably be attributed to the different boundary conditions on the Orr–Sommerfeld operator.

C. Computing the trace of P_{22_2} for Couette flow

We now turn to the evaluation of trace (P_{22_2}), which we remind the reader is given by the following series:

$$\text{trace}(P_{22_2}) = -\frac{1}{2}\sum_{n=1}^{\infty} \langle \phi_n, \mathcal{S}^{-1}\mathcal{C}\mathcal{L}^{-1}(\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^*\phi_n\rangle,$$

where $\{\phi_n, \gamma_n\}$ are the eigenfunctions and eigenvalues of the Squire operator given by (19). This series can be simplified due to the relations between the operators $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{C}$. These relations are somewhat simpler for Couette flow than they are for Poiseuille flow, and we only consider Couette flow in this section, though in principle our methods are also applicable to the latter case.

First we make use of the fact that ϕ_n is an eigenfunction of $\mathcal{S}$, and thus the actions of $\mathcal{S}^{-1}$ and $\mathcal{C}^*$ on it are particularly simple

$$\begin{aligned}
\text{trace}(P_{22_2}) &= -\frac{1}{2}\sum_{n=1}^{\infty} \langle \Phi_n, \mathcal{S}^{-1}\mathcal{C}\mathcal{L}^{-1}(\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^*\phi_n\rangle_2 \\
&= -\frac{1}{2}\sum_{n=1}^{\infty} \langle \mathcal{L}^{-1}\mathcal{C}^*\mathcal{S}^{-1}\phi_n,\ (\mathcal{L}+\gamma_n I)^{-1}\mathcal{C}^*\phi_n\rangle_{\text{os}} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty} \left\langle \mathcal{L}^{-1}\left(\frac{-ik_z}{\gamma_n^2}\right)\phi_n, (\mathcal{L}+\gamma_n I)^{-1}\left(\frac{-ik_z}{\gamma_n}\right)\phi_n\right\rangle_{\text{os}} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{k_z^2}{\gamma_n^3}\langle \mathcal{L}^{-1}\phi_n, (\mathcal{L}+\gamma_n I)^{-1}\phi_n\rangle_{\text{os}}.
\end{aligned} \quad (29)$$

The second equality is due to the self-adjointness of $\mathcal{S}^{-1}$ and $\mathcal{L}^{-1}$ with respect to the appropriate inner products. The third equality follows from ϕ_n being an eigenfunction of the operators $\mathcal{S}^{-1} = \Delta^{-1}$ with eigenvalue $1/\gamma_n$, and that for Couette flow $\mathcal{C}^* = -ik_z\Delta^{-1}$.

We now evaluate the individual terms in the series (29). This task is significantly simplified by utilizing the relations between $\mathcal{L}^{-1}$ and $(\mathcal{L}+\gamma_n I)^{-1}$, and by the fact that ϕ_n is an eigenfunction of a "part" of the operator $\mathcal{L} = \Delta^{-1}\Delta^2$. We summarize this in the following lemma.

Lemma 4: Let g and f be functions over the interval $[-1,1]$ *which are the solution to the following two TPBVPs:*

$$\Delta^2 f = \gamma_n \phi_n, \quad f(\pm 1) = f'(\pm 1) = 0,$$

$$(\Delta^2 + \gamma_n \Delta) g = \gamma_n \phi_n, \quad g(\pm 1) = g'(\pm 1) = 0.$$

Then

(1) $\langle \mathcal{L}^{-1}\phi_n, (\mathcal{L}+\gamma_n I)^{-1}\phi_n\rangle_{\text{os}} = \langle \phi_n, g\rangle_2 - \langle \phi_n, f\rangle_2$.

(2) *The inner products can be determined from the TPBVPs by*

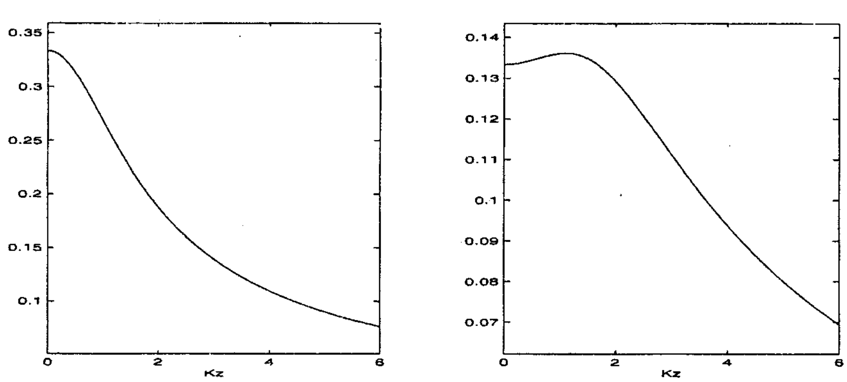

FIG. 2. Trace(P_{22}) (left) and trace(P_{11}) (right) as a function of k_z.

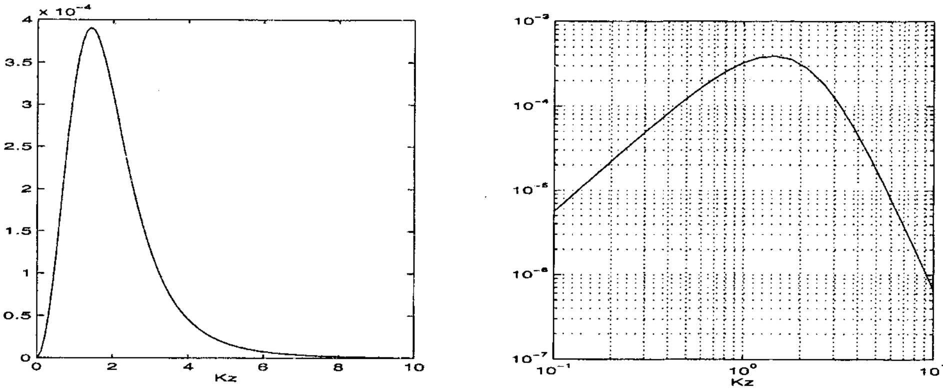

FIG. 3. A regular (left) and logarithmic (right) plot of the coupling term $f_2(k_z)$.

$$\langle \phi_n , f \rangle_2 - \langle \phi_n , g \rangle_2 = \begin{cases} \dfrac{1}{2\gamma_n} + \dfrac{n\pi}{2\gamma_n^2}\left[(f''(1) - f''(-1)) - \dfrac{1}{2}(g''(1) - g''(-1)) \right] & n \text{ even} \\ \dfrac{1}{2\gamma_n} - \dfrac{n\pi}{2\gamma_n^2}\left[(f''(1) - f''(-1)) - \dfrac{1}{2}(g''(1) + g''(-1)) \right] & n \text{ odd} \end{cases}.$$

The expressions in lemma 4, part (2) can be evaluated by solving the two TPBVPs. Since these are constant coefficient ordinary differential equations (ODEs), their solutions are elementary and are given by trigonometric and hyperbolic functions. Once the solutions are found, their second derivatives at $y = \pm 1$ give the series terms as in the previous lemma.

The difficulty is that the ODE coefficients are functions of k_z and n. After some fairly laborious algebraic manipulations and the aid of the MAPLE symbolic computations package, we were able to obtain concise expressions as follows:

$$(-1)^n f''(1) - f''(-1) = \begin{cases} \dfrac{2n\pi}{\gamma_n} \dfrac{k_z(\cosh(2k_z) - 1)}{(\sinh(2k_z) - 2k_z)} & n \text{ even} \\ -\dfrac{2n\pi}{\gamma_n} \dfrac{k_z(\cosh(2k_z) + 1)}{(\sinh(2k_z) + 2k_z)} & n \text{ odd} \end{cases},$$

$$(-1)^n g''(1) - g''(-1) = \begin{cases} \dfrac{n\pi}{2(\alpha_n \coth(\alpha_n) - k_z \coth(k_z))} & n \text{ even} \\ \dfrac{n\pi}{2(\alpha_n \tanh(\alpha_n) - k_z \tanh(k_z))} & n \text{ odd} \end{cases},$$

where $\alpha_n := \sqrt{2k_z^2 + n^2\pi^2/4}$. Putting together all the above-mentioned relevant expressions, we finally obtain the series to express the coupling term as a function of k_z in terms of a series over n:

$$f_2(k_z) = \text{trace}(P_{22_2}) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{k_z^2}{\gamma_n^4} \begin{cases} \left(\dfrac{1}{2} + \dfrac{n^2\pi^2}{2\gamma_n}\left[\dfrac{2k_z(\cosh(2k_z) - 1)}{\gamma_n(\sinh(2k_z) - 2k_z)} - \dfrac{1}{4(\alpha_n \coth(\alpha_n) - k_z \coth(k_z))} \right] \right) & n \text{ even} \\ \left(\dfrac{1}{2} + \dfrac{n^2\pi^2}{2\gamma_n}\left[\dfrac{k_z(\cosh(2k_z) + 1)}{\gamma_n(\sinh(2k_z) + 2k_z)} + \dfrac{1}{4(\alpha_n \tanh(\alpha_n) - k_z \tanh(k_z))} \right] \right) & n \text{ odd} \end{cases}.$$

This function, which is essentially a frequency (k_z) response, is plotted in Fig. 3 as a regular and as a "bode" plot which is common in the frequency analysis of systems. We note that his plot matches plots generated through Galerkin approximations of the original PDE, and is in agreement with those reported in Refs. 9 and 10. The plots show the characteristic peak at $k_z \sim 1.5$, which is the wave number for the most energetic response to stochastic excitation.

V. DISCUSSION

The presence of background noise such as free stream disturbances and wall roughness has long been recognized as an important aspect of transition in boundary layer and channel flows. By modeling such noise as stochastic excitation of linearized channel flow, we have shown that $O(R^3)$ energy amplification is possible for 3-D disturbances. The amplification mechanism is due to coupling between the disturbance's wall-normal velocity and vorticity. This coupling is provided by the mean flow shear and the disturbance's spanwise variation.

We now comment on the role of non-normality in the energy amplification mechanism. There are two reasons for the non-normality of the generator $\mathcal{A}$ in (3). The first being the non-normality of the Orr–Sommerfeld and Squire operators when $k_x \neq 0$. Second, the "off-diagonal" term $\mathcal{C}$ introduces a non-normality easily seen in the two-by-two block representation of $\mathcal{A}$. The latter is much more significant than the former. To see this, we note that in two-dimensional channel flows (i.e., spanwise constant perturbations), the generator of the evolution equation for the stream function is precisely the non-normal $\mathcal{L}$, but in that case only $O(R)$ amplification is possible. On the other hand, in the case of streamwise constant perturbations that we consider in this paper, $k_x = 0$, $\mathcal{L}$ and $\mathcal{S}$ become self-adjoint, yet $O(R^3)$ amplification is possible via the coupling due to the $\mathcal{C}$ term.

Spanwise variations are recognized as leading to three-dimensional structures in transition. Our analysis shows that spanwise varying but streamwise constant perturbations are much more important for transition than spanwise constant and streamwise varying perturbations such as TS waves. Indeed, it has been demonstrated[11] that spanwise modes have substantial growth when even a small amount of background noise (caused by round-off error) is present.

Since the original work on energy growth revealed that streamwise vortices and streaks are the dominant feature in the linearized models rather than TS waves, the question arises as to the relation between these structures and the ubiquitous wall-layer streamwise vortices and streaks in turbulent boundary layers. The streak spacing predicted in the present theory is independent of the Reynolds number, but is dependent on the profile U. The turbulent mean flow profiles in channel flow are different from the corresponding laminar profiles, and are Reynolds number dependent as are the observed coherent structures. The above-mentioned facts suggest the possibility that wall-layer streaky structures could be the result of similar linear amplification processes. However, the quantitative comparison with experimental results needs to be carried out. Further investigations are required to reveal the relevance of this type of analysis to fully turbulent boundary layer flows.

There is a significant conceptual difference between proposing energy growth versus energy amplification as transition mechanisms. Energy growth requires favorably configured initial velocity fields to achieve large transient growth, while energy amplification requires the presence of external forcing to produce perturbed flow fields. Mathematically, it is essentially the same mechanism that causes amplification or growth. Given the extreme sensitivity to shear flows to free stream disturbances and wall roughness, it seems more physically appropriate to think in terms of amplification rather than growth. This is essentially an input–output view where the issue is to quantify the effect of external forcing inputs on flow perturbations viewed as an output.

APPENDIX A: LINEAR SYSTEMS DRIVEN BY STOCHASTIC INPUTS

We summarize the main results of the theory of linear systems with stochastic inputs. With a careful choice of notation, the statements we make are valid for both finite and infinite dimensional systems. Further details can be found in Refs. 21 and 19.

Consider the forced linear system given by

$$\frac{d}{dt}\psi = A\psi + Bd, \tag{A1}$$

where the state $\psi \in \mathbb{R}^n$ in the finite dimensional case, and in some Hilbert space in the infinite dimensional case. The PDE (3) can be viewed as an evolution equation of the above-mentioned form with the state evolving in the Hilbert space $\mathbb{H}_{\text{os}} \oplus L^2$ (since $\hat{v} \in \mathbb{H}_{\text{os}}$ and $\hat{\omega} \in L^2$).

When d is a stochastic process, the notion of a solution to the differential equation (A1) is a little delicate. It is simpler instead to work with the solution in terms of the convolution

$$\psi(t) = e^{tA}\psi(0) + \int_0^t e^{(t-\tau)A} B\, d(\tau)\, d\tau.$$

A second-order stochastic process is given by its correlation statistics. Assuming d is zero mean, its covariance matrix (operator, in the infinite dimensional case) is defined by

$$R(t_1;t_2) := \mathcal{E}\{d(t_1)d^*(t_2)\}.$$

The process is termed wide sense stationary if R is a function of only $t_1 - t_2$. If d is a vector valued process, then d^* is to be interpreted as the complex conjugate transpose of the vector d. In the infinite dimensional case, we can define the above-mentioned product as the tensor produce of d with itself, i.e.,

$$dd^* := d \otimes d,$$

where for any two elements f,g of a Hilbert space, the tensor product is defined as the rank one operator $f \otimes g$ that acts as follows:

$$(f \otimes g)(x) := f\langle g, x\rangle.$$

Note how the Hilbert space inner produce determines the tensor product. The notion of a tensor product is a generalization of the column/row vector product for vectors where the underlying inner produce is the usual Euclidean one. We need to use this slightly more general definition here since the energy form (4) used in this paper is not the standard L^2 inner product. For simplicity, we will use the notation dd^* with the understanding that it stands for the tensor product when d belongs to a Hilbert space.

If d is a stochastic process in (A1), then ψ becomes a process whose second-order statistics are determined by the linear system's dynamics. Assume that d is a temporally stationary, white process, i.e.,

$$\mathcal{E}\{d(t_1)d^*(t_2)\} = \mathcal{R}\delta(t_1 - t_2), \tag{A2}$$

where $\mathcal{R}$ is the "spatial" correlation operator. Assuming zero initial conditions, the correlation operator of ψ can be computed by

$$V(t_1,t_2):=\mathcal{E}\{\psi(t_1)\psi^*(t_2)\}$$
$$=\mathcal{E}\left\{\left(\int_0^{t_1} e^{(t_1-\tau)A}Bd(\tau)d\tau\right)\times\left(\int_0^{t_2} d^*(s)B^*e^{(t_2-s)A^*}ds\right)\right\}$$
$$=\int_0^{\min(t_1,t_2)} e^{(t_1-\tau)A}B\mathcal{R}B^*e^{(t_2-\tau)A^*}d\tau, \quad \text{(A3)}$$

which follows by using the statistics of d (A2). In the steady state limit when $t_1=t_2\rightarrow\infty$, we obtain

$$V=\int_0^{\infty} e^{tA}B\mathcal{R}B^*e^{tA^*}dt. \quad \text{(A4)}$$

We note here that this correlation operator contains all the second-order steady state statistics of the flow field. These can be interpreted as being precisely the Reynolds stresses if the flow is assumed to satisfy the linearized Navier–Stokes equations, and is driven by external forcing with known statistics.

An important fact used in this paper is that the correlation operator can be found without evaluating the integral in (A4). We can show that the steady state V satisfies a so-called Lyapunov equation. Consider expression (A3) at $t_1=t_2=t$, which we will denote by $V(t)$. We can derive an operator differential equation for $V(t)$ by simply differentiating (A3) to obtain

$$\frac{d}{dt}V(t)=AV(t)+V(t)A^*+B\mathcal{R}B^*,$$

the so-called differential Lyapunov equation. If the system is stable, then the steady state limit exists and $\lim_{t\rightarrow\infty}(d/dt)V(t)=0$, which then yields the algebraic Lyapunov equation

$$AV+VA^*+B\mathcal{R}B^*=0.$$

In this paper, we have assumed the external input statistics to be given by $\mathcal{R}=I$ (i.e., a spatially uncorrelated process), and since the forcing enters directly into the state equation, $B=I$, which then yields Eq. (5).

We note that in the controls and systems theory literature, one often encounters Lyapunov equations. They are linear equations in terms of the unknown matrix (or operator) V. In the case of finite dimensional systems, there exist efficient algorithms for solving the resulting linear system of equations. For infinite dimensional systems the entries are operators, and except in simple cases, it is not possible to explicitly solve the equations without resorting to finite dimensional approximations. In our study, we are able to circumvent the approximations since we require only the operator trace of the solution rather than the full operator.

Finally, we comment that the variance amplification captured by tr(V) is precisely what is referred to in the controls literature as the $\mathcal{H}^2$ norm of the system from d to ψ. This norm can be related to the transient response by (here we assume $B=\mathcal{R}=I$)

$$\text{tr}(V)=\text{tr}\left(\int_0^{\infty} e^{tA}e^{tA^*}dt\right)=\int_0^{\infty}\text{tr}(e^{tA}e^{tA^*})dt.$$

The numerical investigations of energy growth[2,4] revealed $O(R^2)$ peak values for $\lambda_{\max}(e^{tA}e^{tA^*})$, with the peak occurring at time $O(R)$. Since the energy amplification is the integral, one can intuitively expect tr(V) to be $O(R^3)$. This, however, may not always be the case, since for any operator T, tr(T) can in general be quite different from $\lambda_{\max}(T)$.

APPENDIX B: PROOF OF LEMMA 4

Proof: Part 1. The first part follows from the properties of $\mathcal{L}$. *Let us define*

$$f:=\mathcal{L}^{-1}\phi_n\Rightarrow\mathcal{L}f=\phi_n\Rightarrow\Delta^2 f=\Delta\phi_n=\gamma_n\phi_n,$$
$$g:=(\mathcal{L}+\gamma_n I)^{-1}\phi_n\Rightarrow\mathcal{L}g+\gamma_n g=\phi_n\Rightarrow\Delta^2 g$$
$$=-\gamma_n\Delta g+\Delta\phi_n\Rightarrow(\Delta^2+\gamma_n\Delta)g=\gamma_{n\phi_n}.$$

Note that the boundary conditions of f and g follow from the definition of the operator $\mathcal{L}$. *The inner product then becomes*

$$\langle\mathcal{L}^{-1}\phi_n,(\mathcal{L}+\gamma_n I)^{-1}\phi_n\rangle_{\text{os}}$$
$$=\langle f,g\rangle_{\text{os}}=-\langle f,\Delta g\rangle_2$$
$$=-\left\langle f,\frac{-1}{\gamma_n}\Delta^2 g+\phi_n\right\rangle_2$$
$$=\frac{1}{\gamma_n}\langle\Delta^2 f,g\rangle_2-\langle f,\phi_n\rangle_2$$
$$=\frac{1}{\gamma_n}\langle\Delta\phi_n,g\rangle_2-\langle f,\phi_n\rangle_2$$
$$=\langle\phi,g\rangle_2-\langle f,\phi_n\rangle_2.$$

Note that we made use of the identity $\langle f,\Delta^2 g\rangle_2=(\Delta^2 f,g\rangle_2$ *which is valid because of the boundary conditions on f and g.*

Part 2. The key here is the following general property of the Δ^2 *operator*:

$$\langle\Delta^2 h_1,h_2\rangle_2=\langle h_1,\Delta^2 h_2\rangle_2+((h_1'''h_2-h_1''h_2'+h_1'h_2''-h_1h_2''')-2k_z^2(h_1'h_2-h_1h_2'))|_{-1}^{1},$$

which can be verified through integration by parts. Recalling that $\phi_n(y)=\sin((n\pi/2)(y+1))$ *is an eigenfunction of* Δ^2, we compute

$$\langle\phi_n,f\rangle_2=\frac{1}{\gamma_n^2}\langle\Delta^2\phi_n,f\rangle_2$$
$$=\frac{1}{\gamma_n^2}[\langle\phi_n,\Delta^2 f\rangle_2+(\phi_n'f'')|_{-1}^{1}]$$
$$=\frac{1}{\gamma_n^2}\left[\langle\phi_n,\gamma_n\phi_n\rangle_2+\frac{n\pi}{2}(\cos(n\pi)f''(1)-\cos(0)f''(-1))\right]$$
$$=\left[\frac{1}{\gamma_n}+\frac{n\pi}{2\gamma_n^2}((-1)^n f''(1)-f''(-1))\right].$$

The evaluation of the second inner product proceeds along similar lines using the TPBVP defining g. Omitting the details, we obtain the final result

$$\langle \phi_n , g \rangle_2 = \left[\frac{1}{2\gamma_n} + \frac{n\pi}{4\gamma_n^2} ((-1)^n g''(1) - g''(-1)) \right],$$

which, when combined with the previous expression, proves part 2.

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