

Myths, Misconceptions and Misuses of Nonlinearity

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The Issues

Observed Phenomena

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Mathematical Models

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Mathematical Models

- Phenomena: Data, system trajectories, time traces
Models: Differential equations

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- Phenomena: Data, system trajectories, time traces
Models: Differential equations
- Concern: **Complexity**
 - Complexity of phenomena (scale, patterns, etc.)
somewhat subjective

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Mathematical Models

- Phenomena: Data, system trajectories, time traces
Models: Differential equations

- Concern: **Complexity**

- Complexity of phenomena (scale, patterns, etc.)

somewhat subjective

- Complexity of models (?????)

- Linear vs. Nonlinear

- Low vs. High order

-

Outline

- A story of boundary layer turbulence

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 - Effective analysis using tools from linear systems theory

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 - Linearization techniques
 - Model order
 - Special structure

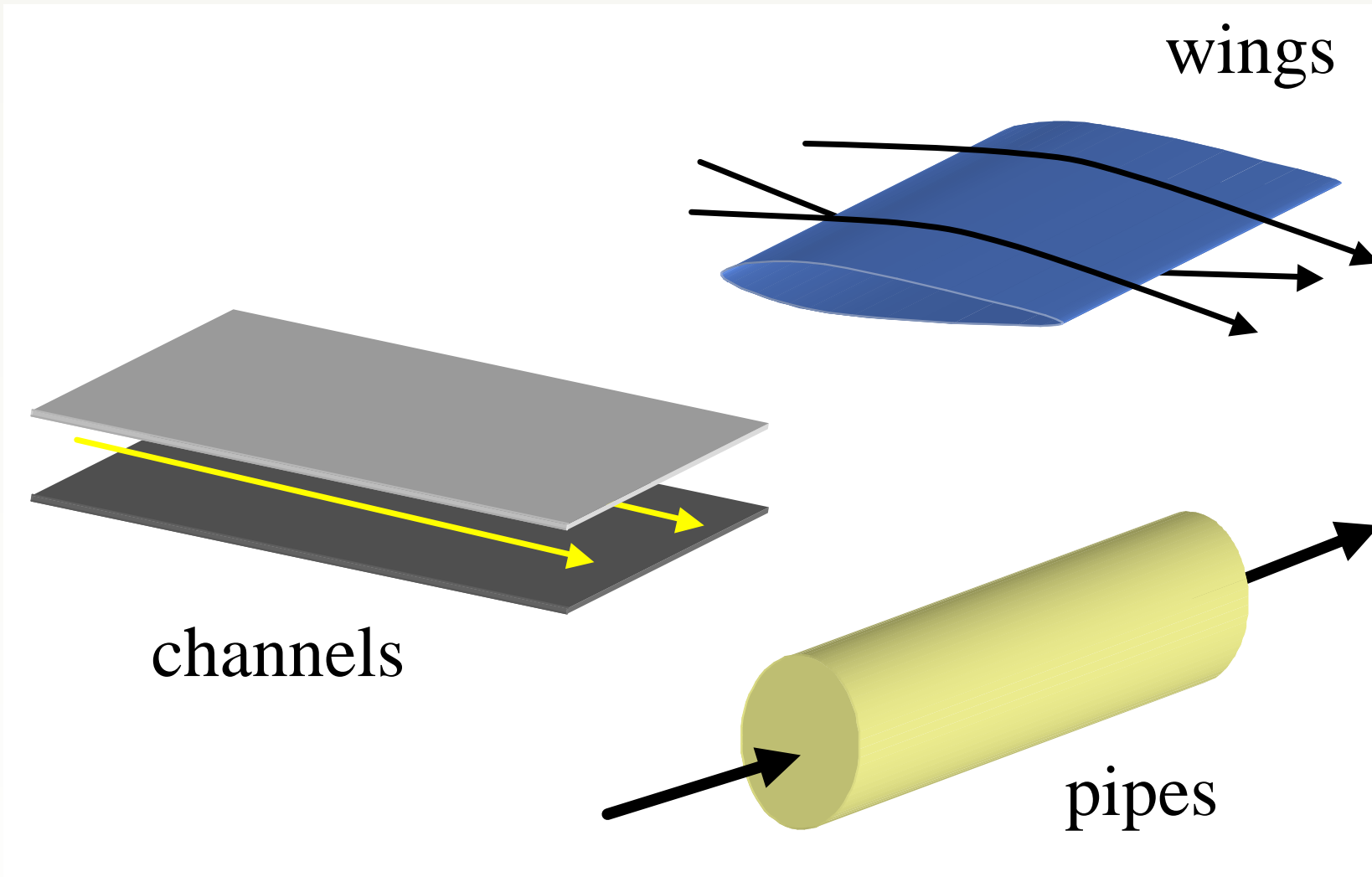
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- A story of boundary layer turbulence
 - Effective analysis using tools from linear systems theory
 - “but I thought turbulence was a nonlinear phenomenon!!??”
- Are linear and nonlinear models equivalent?
 - Linearization techniques
 - Model order
 - Special structure
- The use of “nonlinearity” in current scientific culture
 - a runaway concept

The phenomenon of turbulence

Technologically important flows:

Flows past *streamlined* bodies



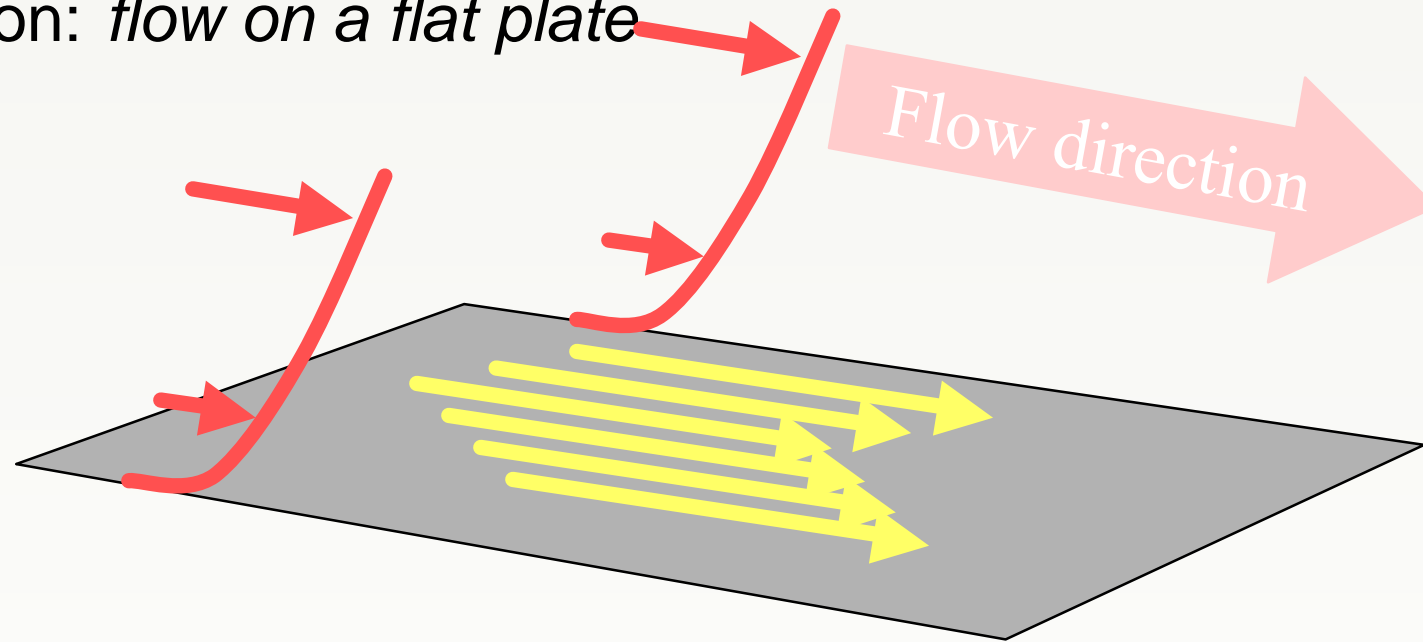
Wall-bounded shear flows

Friction with the walls drives the flows

The phenomenon of turbulence (Cont.)

Boundary layers form in flow past any surface

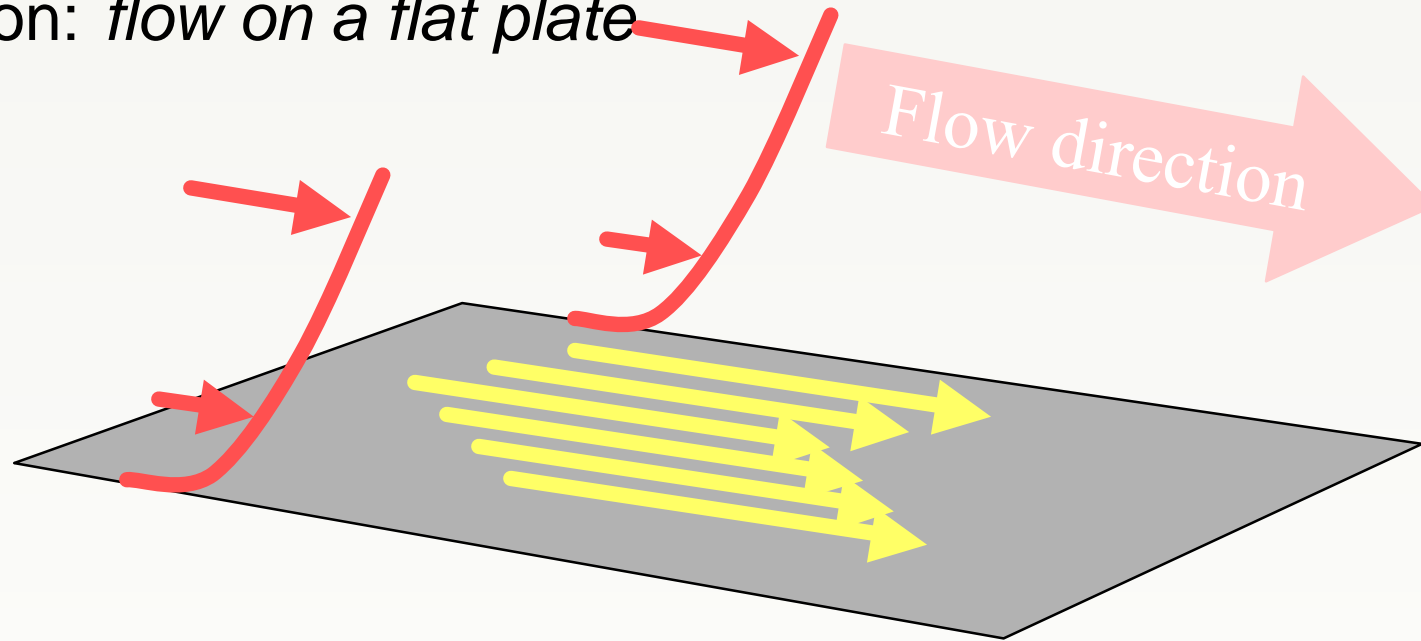
Idealization: *flow on a flat plate*



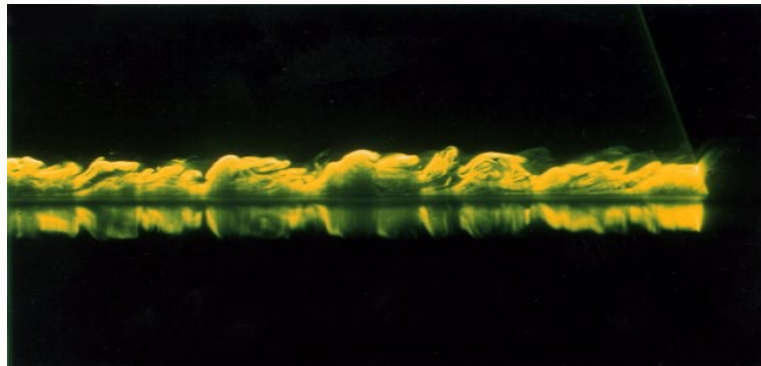
The phenomenon of turbulence (Cont.)

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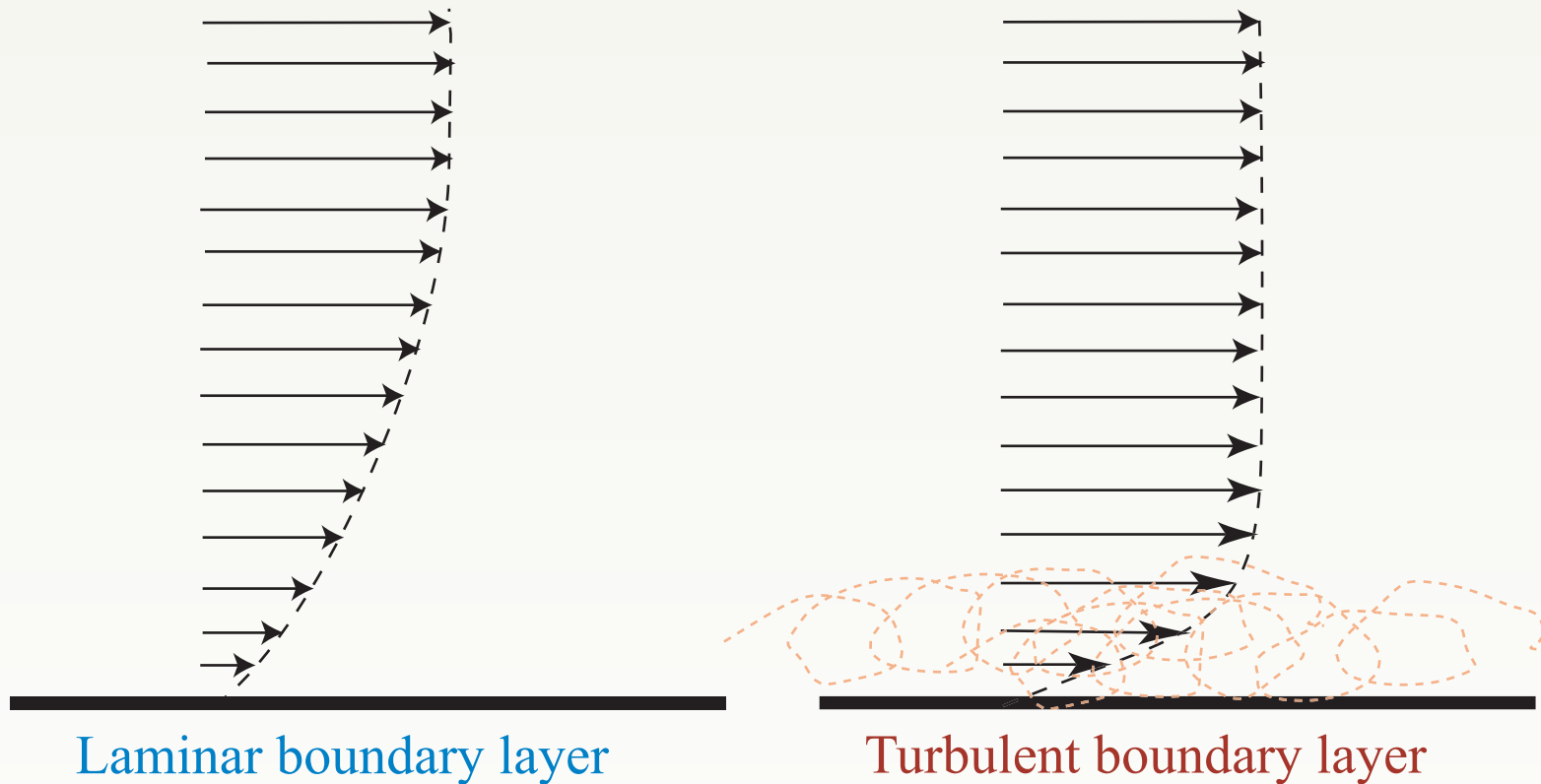
Idealization: *flow on a flat plate*



Viewed sideways

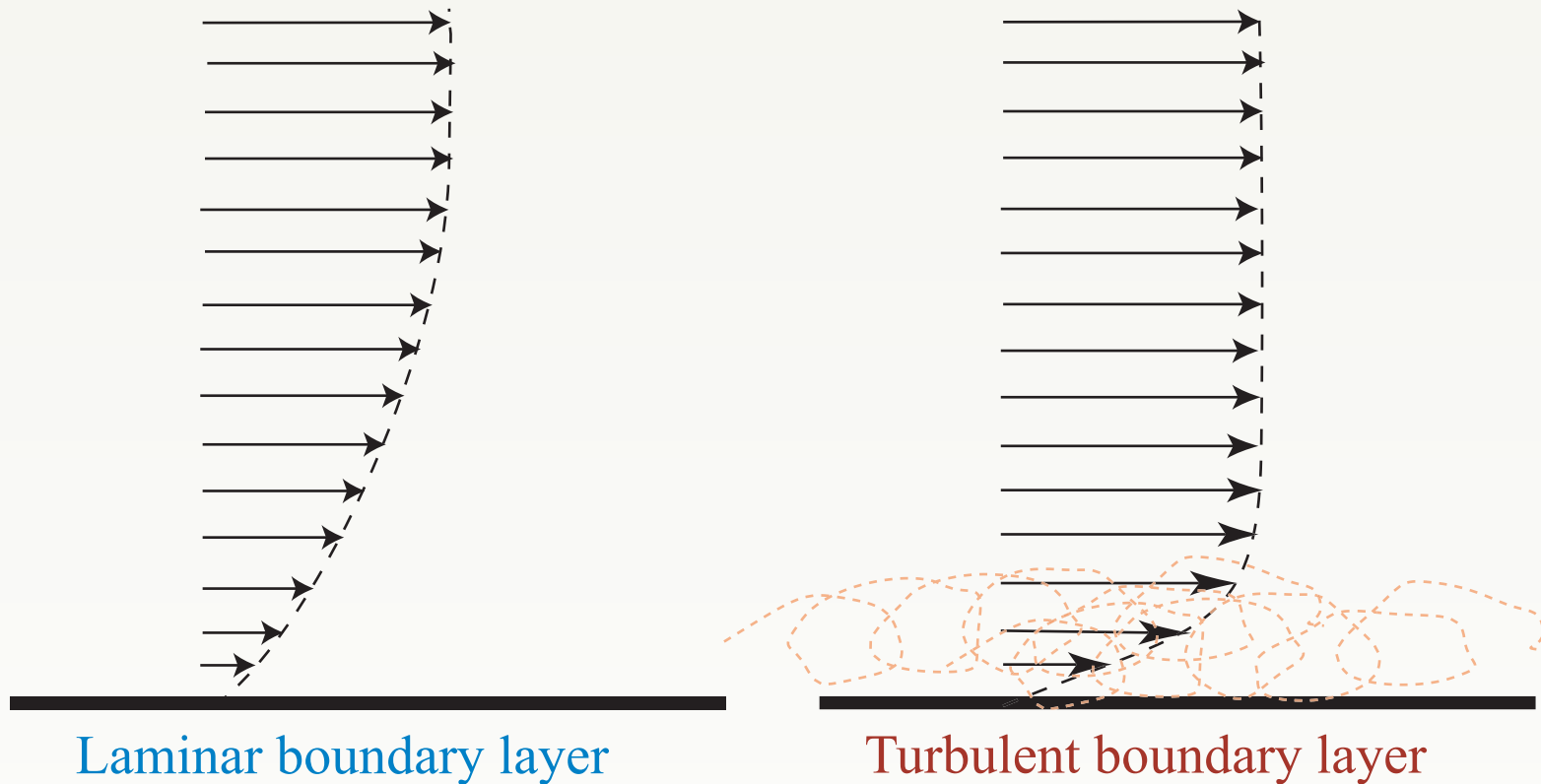


Boundary layer turbulence and skin-friction drag



A laminar BL causes less drag than a turbulent BL (for same free-stream velocity)

Boundary layer turbulence and skin-friction drag



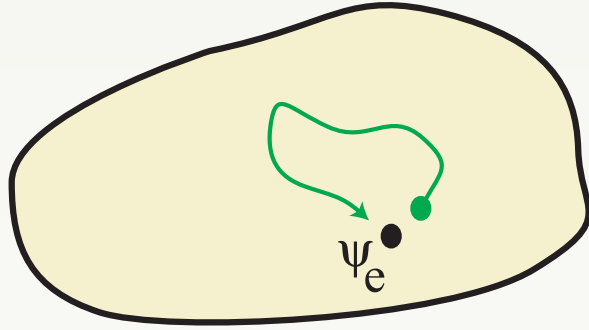
A laminar BL causes less drag than a turbulent BL (for same free-stream velocity)

This *skin-friction drag* is 40-50% of total drag on typical airliner

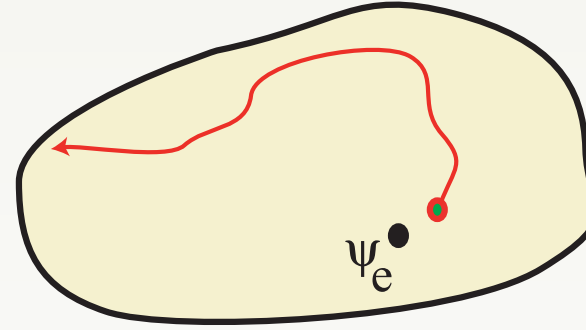


Does the system stay near an equilibrium?

If starting “near” equilibrium, does system come back to it??



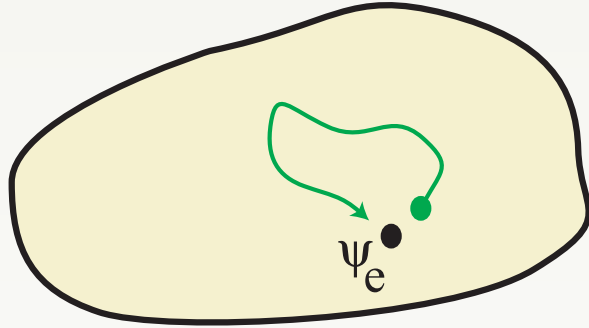
stable equilibrium



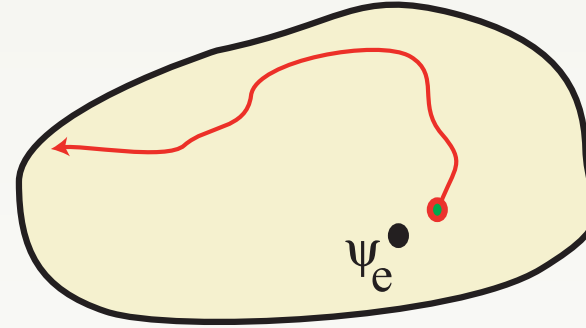
unstable equilibrium

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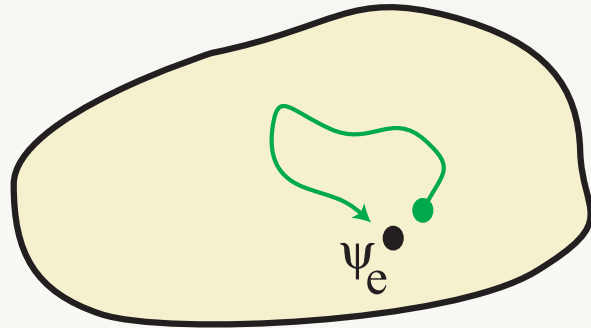


unstable equilibrium

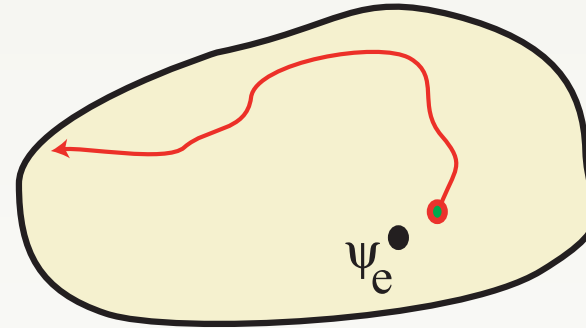
An unstable equilibrium is not really an “equilibrium”

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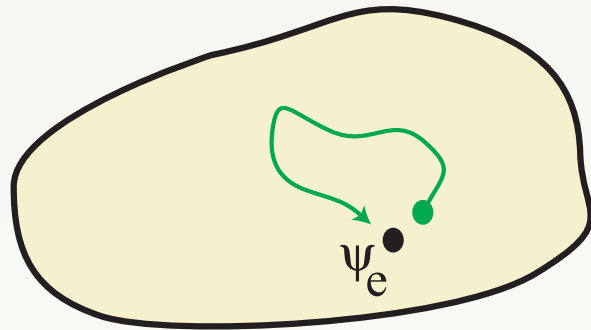
How to check stability?

Common method: Linearization

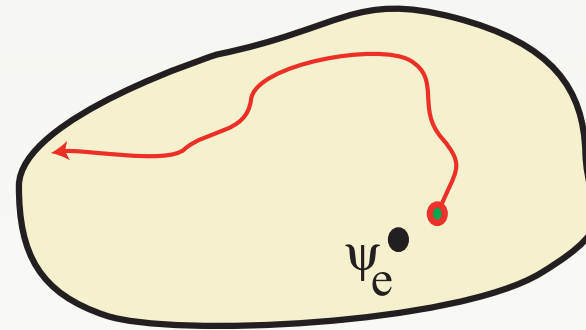
$$\partial_t \Psi(t) = \mathcal{F}(\Psi(t)) = \underset{\substack{\uparrow \\ \text{Linear part}}}{\mathcal{A}(\Psi(t))} + \underset{\substack{\uparrow \\ \text{The rest}}}{\mathcal{N}(\Psi(t))}$$

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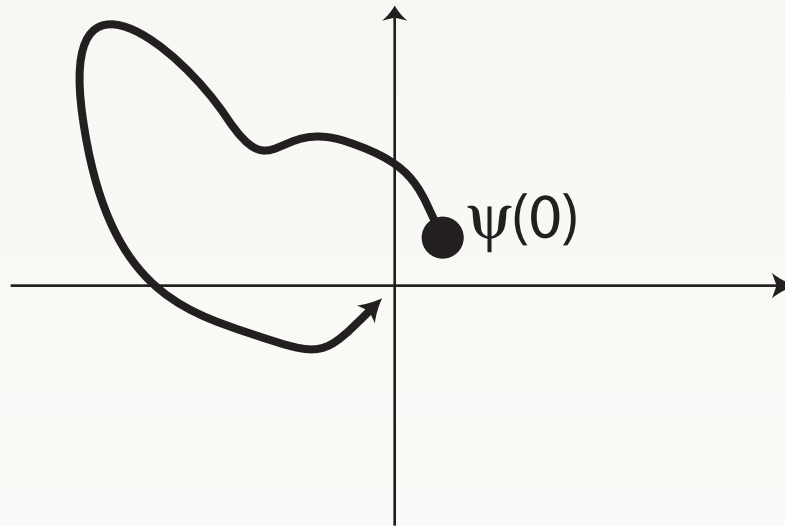
Stability (instability) of \mathcal{A} \Leftrightarrow Local stability (instability) of \mathcal{F}

Can be studied using eigenvalue/eigenfunction analysis of \mathcal{A}

Uncertainty in a Dynamical System

Lyapunov Stability

deals with **uncertainty** in initial conditions



If $\Psi(0)$ is known to be *precisely* Ψ_e ,

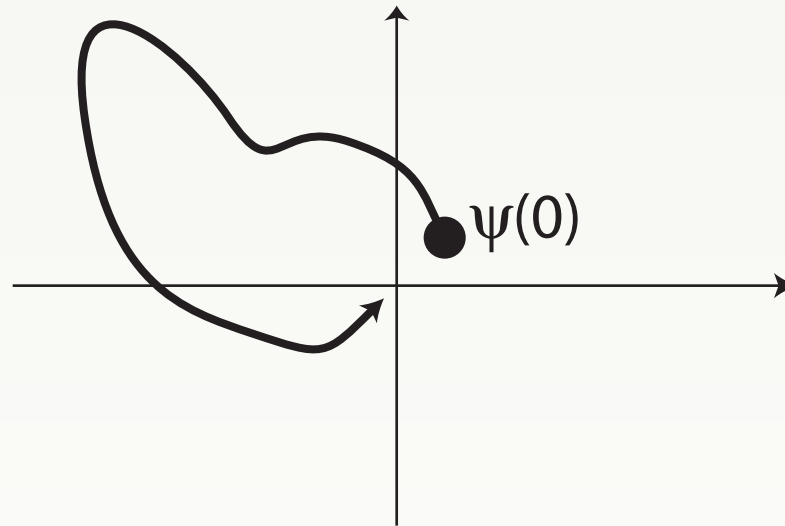
then $\Psi(t) = \Psi_e, t \geq 0$

We introduce the concept of Lyapunov stability because we can never be *infinitely certain* about the initial condition

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- Shortcomings of Lyapunov stability** {
- Perturbs only initial conditions
 - Cares only about asymptotic behavior

Uncertainty in a Dynamical System (cont.)

Lyapunov stability

$$\dot{\psi} = f(\psi)$$

uncertain initial

conditions

investigate $\lim_{t \rightarrow \infty} \psi(t)$

Uncertainty in a Dynamical System (cont.)

investigate transients

e.g. $\sup_{t \geq 0} \|\psi(t)\|$

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$$\dot{\psi} = F(\psi) + \Delta(\psi)$$

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$$\dot{\psi}(t) = F(\psi(t), d(t))$$

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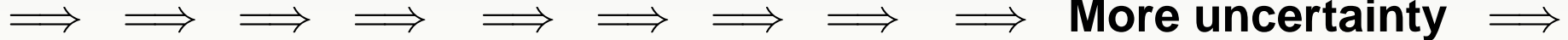
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combinations

$$\dot{\psi}(t) = F(\psi(t), d(t)) + \Delta(\psi(t), d(t))$$

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Uncertainty in a Dynamical System (cont.)

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$\implies \implies \implies \implies \implies \implies \implies \implies \implies \implies$ **More uncertainty** \implies

Linearized version:

transient growth

eigenvalue stability

$$\dot{\psi} = A\psi$$

unmodelled dynamics

$$\dot{\psi} = (A + \Delta)\psi$$

Pseudo-spectrum

combinations

$$\dot{\psi}(t) = (A + B\Delta C)\psi(t) + (F + G\Delta H)d(t)$$

exogenous disturbances

$$\dot{\psi}(t) = A\psi(t) + Bd(t)$$

input-output analysis

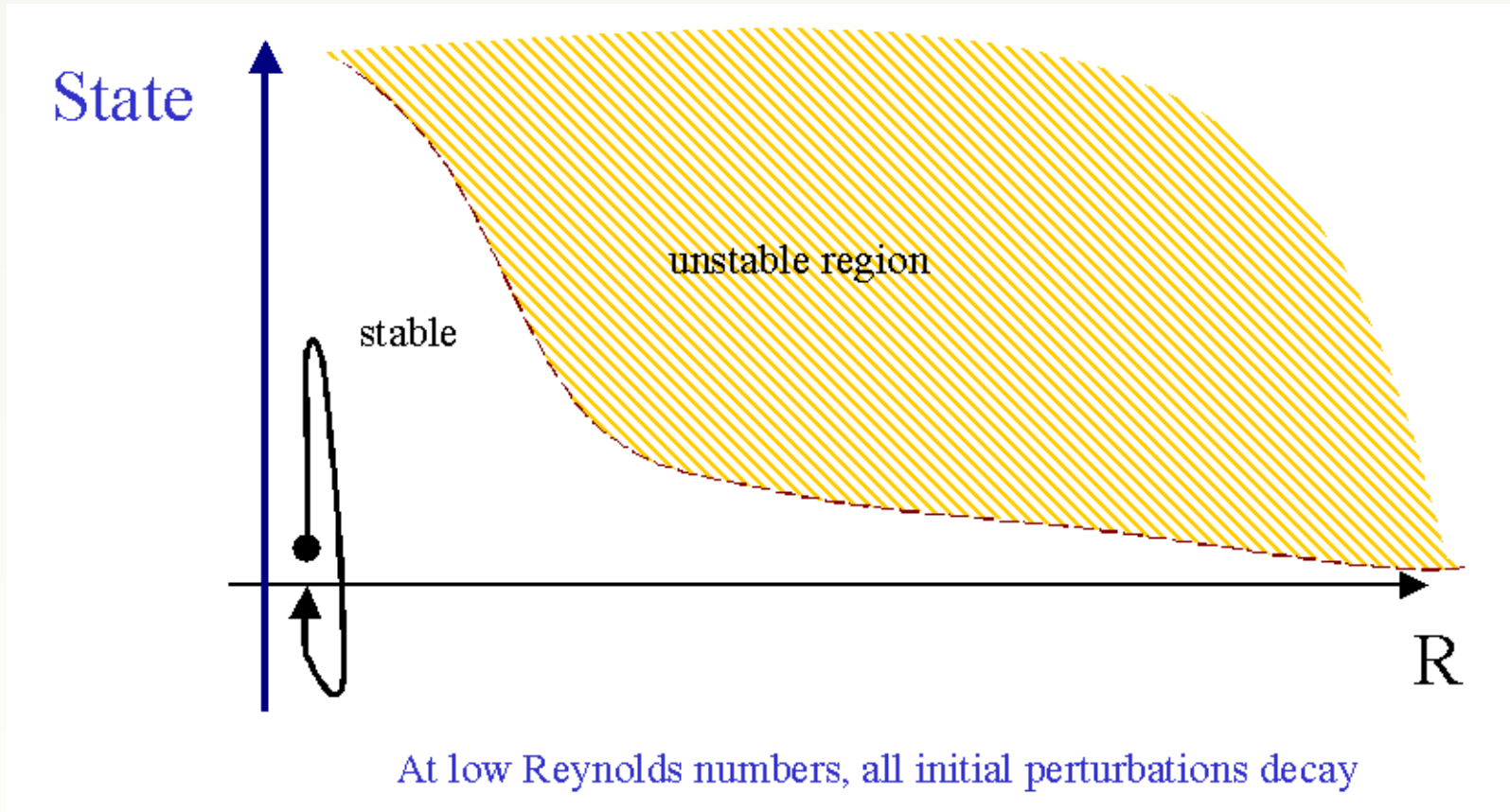
The nature of turbulence

Fluid dynamics are described by deterministic equations

Why does fluid flow “look random” at high Reynolds numbers??

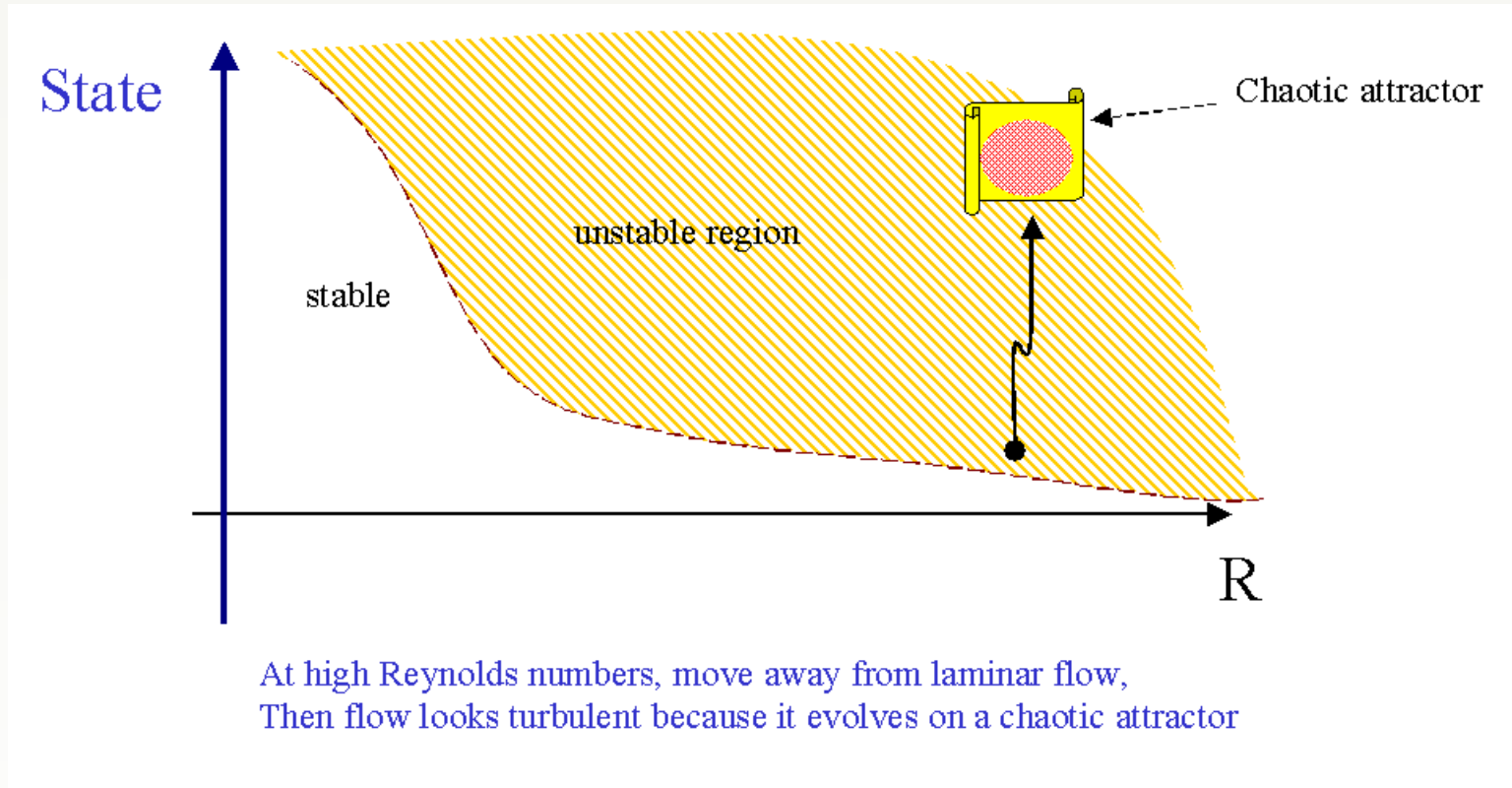
The nature of turbulence

Common view of turbulence



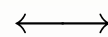
The nature of turbulence (cont.)

Common view of turbulence



Intuitive reasoning

Complex, statistical looking behavior



System with chaotic dynamics

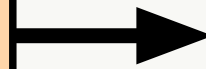
The nature of turbulence (cont.)

Alternative view

Noise
Surface roughness
Thermal forces
Free stream disturbances



**Dynamics of the
Navier-Stokes
Equations**



Fluctuating
flow field
(looks statistical)

The nature of turbulence (cont.)

Alternative view

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Surface roughness
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**Dynamics of the
Navier-Stokes
Equations**



Fluctuating
flow field
(looks statistical)

Qualitatively similar to

Noise
Surface roughness
Thermal forces
Free stream disturbances



**Dynamics of the
Linearized
Navier-Stokes
Equations**



Fluctuating
flow field
(looks statistical)

So what now?

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- Are linear and nonlinear models equivalent?
 - Linearization techniques
 - Model order
 - Special structure
- The use of “nonlinearity” in current scientific culture
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Linearization techniques

The Carleman Linearization

Example: $\frac{d}{dt}x = x^2$

Define: $x_1 := x, x_2 := x^2, \dots, x_n := x^n, \dots$

Linearization techniques

The Carleman Linearization

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Define: $x_1 := x, x_2 := x^2, \dots, x_n := x^n, \dots$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4 & \dots \\ \vdots & & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}, \quad \text{i.e.} \quad \dot{\mathcal{X}} = \mathcal{A}\mathcal{X}$$

The original system is imbedded in this linear system

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General procedure: Given $\dot{x} = F(x, u)$

Define \mathcal{X} with components $x^i u^j$

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + (i) \mathcal{B}\mathcal{X}$$

Thus: *Bilinear Systems are Universal Models*

Linearization techniques (cont.)

- **The Lie-Koopman Linearization**

Linearization techniques (cont.)

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- **State transformations and feedback linearization**

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- **The Fokker-Planck equation**

Linearization techniques (cont.)

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- **State transformations and feedback linearization**
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-

Linearization techniques often do not make a problem more tractable

e.g. computing e^{tA} can be arbitrarily complex!!

Linear Phenomena

Q: What phenomena can a linear dynamical system explain?

$$\frac{d}{dt}\Psi(t) = \mathcal{A} \Psi(t)$$

A: Any phenomena explainable using a nonlinear dynamical system

Linear Phenomena

Q: What phenomena can a linear dynamical system explain?

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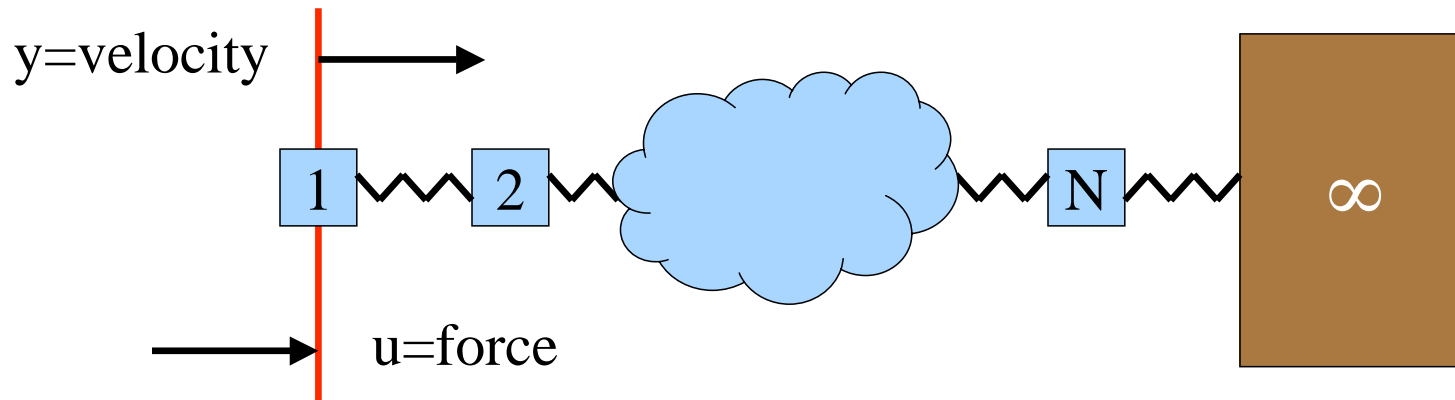
A: Any phenomena explainable using a nonlinear dynamical system

The term “nonlinear phenomenon” has no mathematical meaning!

“Chaos” in Linear Systems

Homogeneous N masses, $N+1$ springs

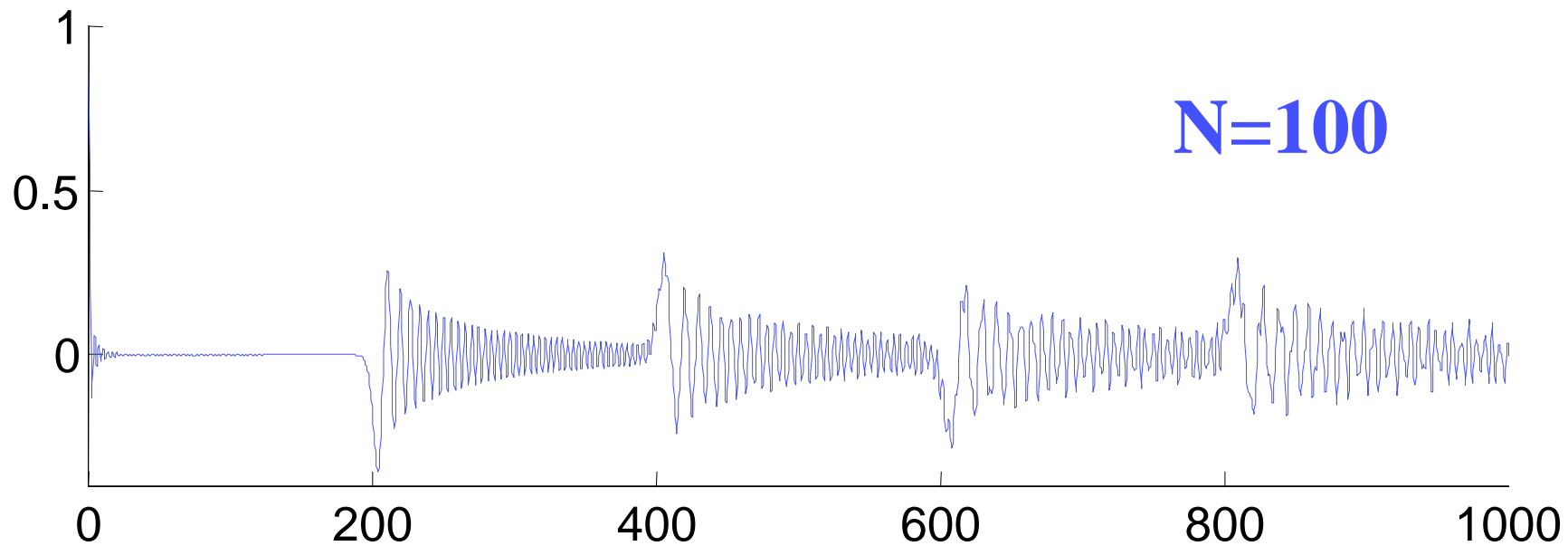
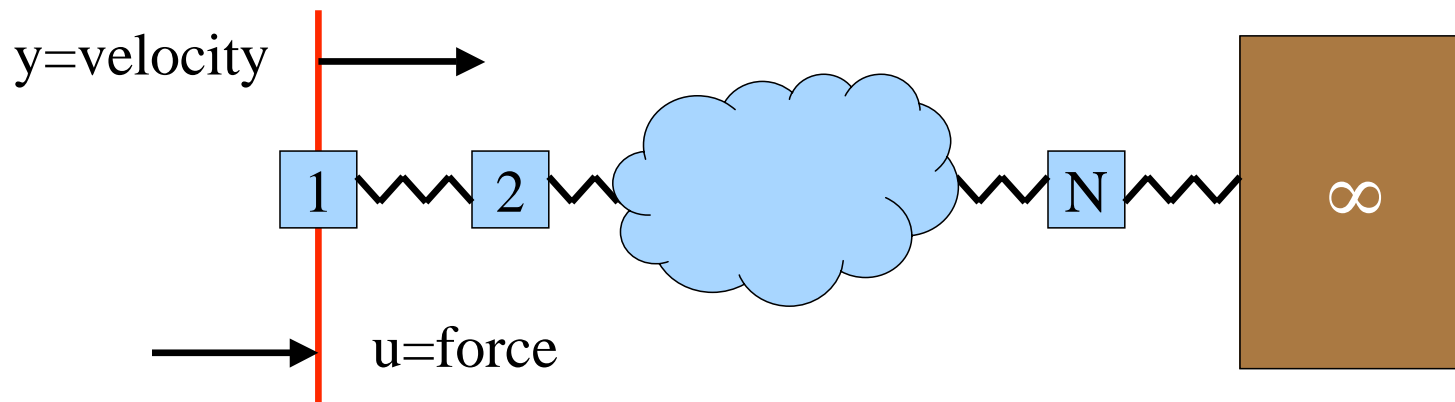
Mass=1
Spring $K=1$



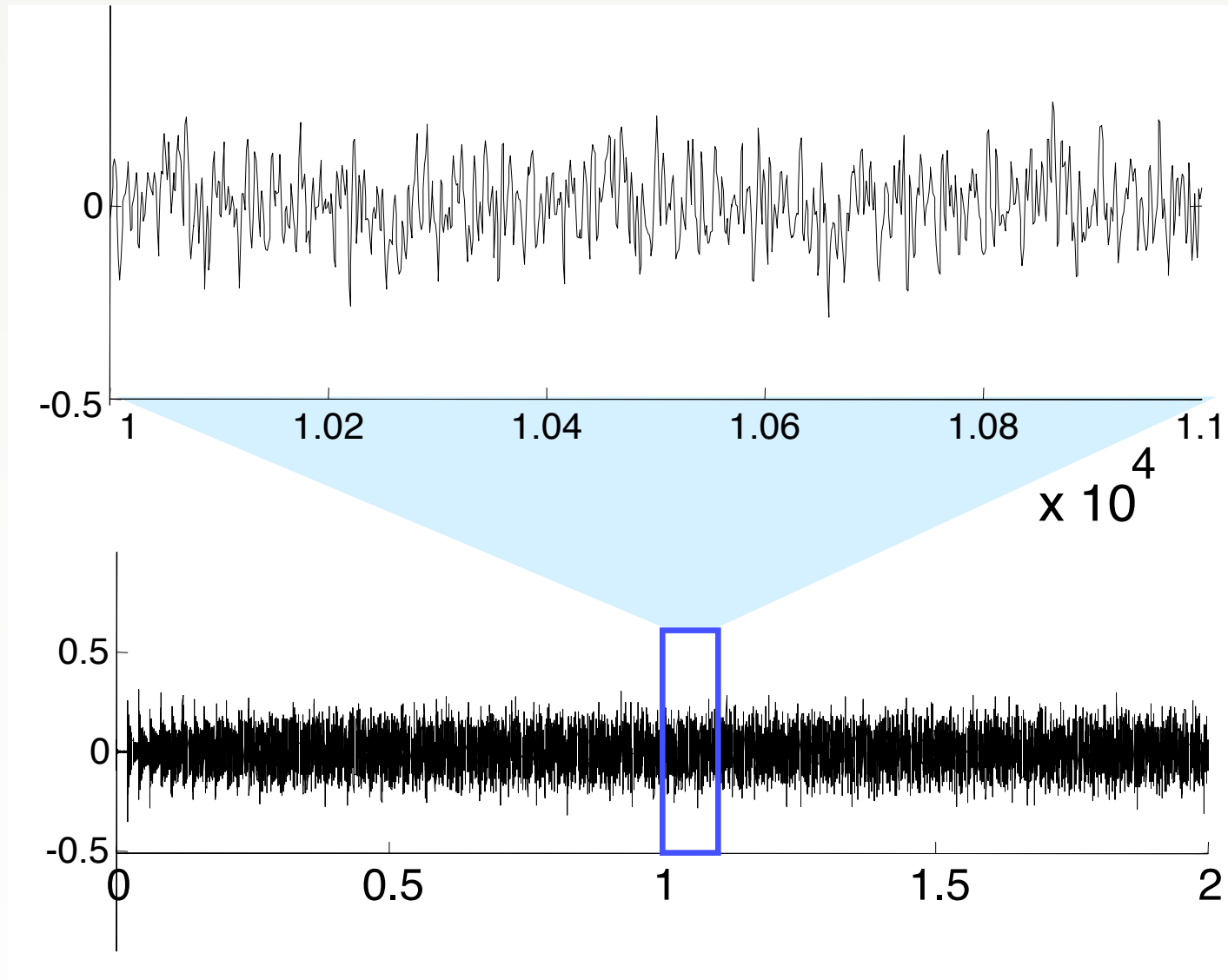
“Chaos” in Linear Systems

Homogeneous N masses, $N+1$ springs

Mass=1
Spring $K=1$



“Chaos” in Linear Systems (cont.)



has continuous spectrum (PSD) over long, but finite times

“Chaos” in Linear Systems (cont.)

- With similar schemes,
can generate stochastic processes with any prescribed PSD
- Clearly not the most efficient method to generate such processes!

Moral: *Sometimes a low dimensional nonlinear model is easier to handle than a high dimensional linear one*

“Chaos” in Linear Systems (cont.)

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can generate stochastic processes with any prescribed PSD
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Moral: *Sometimes a low dimensional nonlinear model is easier to handle than a high dimensional linear one*

Another example: *Optimal filtering equations*

Phrases to avoid

A well defined mathematical object: **nonlinear mapping**

mapping between two vector spaces that does not preserve their linear structure

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- Even the term “nonlinear system” is a little ambiguous

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To make sure you are on firm ground:

use the term nonlinear only in its original mathematical meaning

The End