

Parametric resonance in spatially distributed systems

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Abstract

We consider spatially distributed systems described by Partial Differential Equations (PDEs) in which some of the coefficients are spatially periodic functions. Such systems arise in certain distributed sensorless control schemes which we term spatio-temporal vibrational control, which is a generalized version of standard temporal vibrational control. The mechanism by which certain sensorless periodic feedbacks stabilize or destabilize systems is more generally known as parametric resonance. We develop a spatio-temporal lifting framework using which we analyze stability and system norms of PDEs with periodic coefficients. Examples of PDEs in which parametric resonance occurs are given.

1 Introduction

As is well known, when certain Linear Time Invariant (LTI) systems are connected in feedback with temporally periodic gains, their stability properties can be altered. Depending on the relation between the period of the time varying gain and the dominant modes of the LTI system, the overall system may be stabilized or destabilized. This phenomenon is known in the general literature as *parametric resonance*, the prime example of which is the Mathieu equation. In the controls literature, this phenomenon has been used in sensorless feedback schemes to stabilize unstable systems, and is normally referred to as *vibrational control*.

In this paper we investigate similar parametric resonance mechanisms for spatially distributed systems described by PDEs with periodic coefficients. Our aim is twofold: first to characterize exponential stability of the distributed system, and then to characterize input-output system norms. To this end, we develop a lifting technique similar to the temporal lifting technique for linear periodic Ordinary Differential Equations (ODEs) [1]. This technique provides for a strong equivalence between distributed spatially periodic systems defined on a continuous spatial domain with *spatially invariant* systems defined on a discrete lattice.

For the purpose of stability characterization, the lifting technique is in some sense equivalent to the more widely used Floquet analysis of periodic PDEs. However, Floquet analysis does not easily lend itself to the computation of system norms and sensitivities. We show how the lifting technique can be used to compute both the spectrum of the generating operator and \mathcal{H}_∞ and \mathcal{H}_2

norms of distributed systems.

Our presentation is organized as follows: we begin next section with some mathematical preliminaries regarding the spatial lifting technique. We then present a convenient representation of lifted transfer functions using particular basis sets developed in [2] for use with periodic temporal systems. We then show how to write down a representation of the lifted systems in the transform domain simply from the Fourier symbol of the original PDE operator and the Fourier expansion series of the periodic gains. Finally, we present some examples of PDEs which exhibit parametric resonance, and use the lifting technique to calculate regions of stability and instability as functions of the period and amplitude of the periodic gains.

2 Preliminaries

We consider distributed systems of the form

$$\partial_t \psi = \mathcal{A}\psi + \mathcal{B}u, \quad (1)$$

where \mathcal{A} is a linear operator with periodic coefficients defined on a dense domain $\mathcal{D}(\mathcal{A}) \subset L^2(-\infty, \infty)$, \mathcal{B} is an input operator, u is a forcing term, and ψ is a field of interest determined by the solution of the above equation. Our objective is to investigate dynamical properties of system (1) by computing the operator \mathcal{A} eigenvalues and input-output norms.

For X -periodic system (1), the ‘block-Toeplitz’ decomposition of operator \mathcal{A} is determined by $\bar{\mathcal{A}}_k = \Pi_{L^2[kX, (k+1)X]} \mathcal{A} \Big|_{L^2[0, X] \cap \mathcal{D}(\mathcal{A})}$, where $\Pi_{L^2[kX, (k+1)X]}$ represents the orthogonal projection on $L^2[kX, (k+1)X]$, and $L^2(-\infty, \infty)$ is decomposed into $L^2(-\infty, \infty) = \cdots \oplus L^2[-X, 0] \oplus L^2[0, X] \oplus \cdots$.

The difficulty with finding $\bar{\mathcal{A}}_k$ is related to the issues of its appropriate domain. We now illustrate how this problem can be circumvented. Since \mathcal{A} has periodic coefficients it commutes with T_X , the operator of translation by X , $(T_X \psi)(x) := \psi(x - X)$. Therefore the operator semigroup $G(t) := e^{\mathcal{A}t}$ satisfies $G(t)T_X = T_X G(t)$, i.e. $G(t)$ is also ‘periodic’ even though it usually is not a differential operator. Thus, we can now define the ‘block-Toeplitz’ decomposition of $G(t)$ as $\bar{G}_k(t) := \Pi_{L^2[kX, (k+1)X]} G(t) \Big|_{L^2[0, X]}$. Since $G(t)$ is a bounded operator, indeed there are no issues of domain.

Furthermore, it follows that if $\{\bar{f}_k\}$ denotes the lifting to $L^2_{L^2[0, X]}$ of any $f \in L^2(-\infty, \infty)$ [1], that is, $\bar{f}_k(\hat{x}) := f(kX + \hat{x})$, $k \in \mathbb{Z}$, $0 \leq \hat{x} \leq X$, then $g = G(t)f \Rightarrow \bar{g}_i = \sum_{j \in \mathbb{Z}} \bar{G}_{i-j}(t) \bar{f}_j$. One might think of defining the \bar{A}_k as $\bar{A}_k := \lim_{t \rightarrow 0^+} \frac{1}{t} (\bar{G}_k(t) - I)$, but this is problematic since $\{\bar{G}_k(t)\}$, as one can show, is not a semigroup even though $\{G(t)\}$ is.

However, it is much easier to work with the z -transform evaluated on the unit circle, $z = e^{j\theta}$; $j := \sqrt{-1}$; $\theta \in [0, 2\pi)$, by defining

$$\hat{f}_\theta := \sum_{k \in \mathbb{Z}} \bar{f}_k e^{-j\theta k}, \quad \bar{f}_k \in L^2_{L^2[0, X]},$$

and

$$\hat{G}_\theta(t) := \sum_{k \in \mathbb{Z}} \bar{G}_k(t) e^{-j\theta k}.$$

Now for each $\theta \in [0, 2\pi)$, it can be shown that $\hat{G}_\theta(t)$ are indeed semigroups. The reason for this is a well known convolution property of z -transform which says that if $G_1, G_2, G_3 : L^2(-\infty, \infty) \rightarrow L^2(-\infty, \infty)$ then

$$G_1 = G_2 G_3 \Leftrightarrow \hat{G}_{1\theta} = \hat{G}_{2\theta} \hat{G}_{3\theta}, \quad \forall \theta \in [0, 2\pi).$$

Therefore,

$$\begin{aligned} G(t_1 + t_2) &= G(t_1)G(t_2) \Rightarrow \\ \hat{G}_\theta(t_1 + t_2) &= \hat{G}_\theta(t_1)\hat{G}_\theta(t_2), \quad \forall \theta \in [0, 2\pi). \end{aligned}$$

Since $\hat{G}_\theta(t)$ is a semigroup for each $\theta \in [0, 2\pi)$, we can now define \hat{A}_θ as

$$\hat{A}_\theta := \lim_{t \rightarrow 0^+} \frac{1}{t} (\hat{G}_\theta(t) - I), \quad (2)$$

which transforms $\partial_t \psi = \mathcal{A}\psi$ into $\partial_t \hat{\psi}_\theta(t) = \hat{A}_\theta \hat{\psi}_\theta(t)$. Note that for each $\theta \in [0, 2\pi)$, $\hat{A}_\theta : L^2[0, X] \supset \mathcal{D}(\hat{A}_\theta) \rightarrow L^2[0, X]$ is typically an unbounded operator whose domain can be calculated from (2). In this paper we illustrate that computing \hat{A}_θ can actually be done rather explicitly and simply for PDE operators with periodic coefficients.

As aforementioned, for every $f \in L^2(-\infty, \infty)$ the lifted signal can be defined as $\bar{f}_k(\hat{x}) := f(kX + \hat{x})$, $k \in \mathbb{Z}$, $0 \leq \hat{x} \leq X$. The lifting can be visualized as a breaking a continuous-time signal up into a sequence of pieces which take their values in a functional space of a length X . In other words, $\{\bar{f}_k\} \in L^2_{L^2[0, X]}$, i.e. for any given $k \in \mathbb{Z}$, $\bar{f}_k \in L^2[0, X]$. The lifting operator W_X is one-to-one and onto and thus has a well defined inverse given by $f = W_X^{-1} \bar{g}$, $f(x) = \bar{g}_k(x - kX)$, $k \in \mathbb{Z}$, $kX \leq x \leq (k+1)X$. The action of the operator W_X^{-1} is exactly the opposite of the one caused by the lifting operator. Namely, W_X^{-1} puts together a sequence of function pieces whose value belongs to $L^2[0, X]$, giving a function $f \in L^2(-\infty, \infty)$ [1].

The z -transform of the sequence of signals $\{\bar{f}_k\}$ evaluated on the unit circle is determined by $\hat{f}_\theta(\hat{x}) := \sum_{k \in \mathbb{Z}} \bar{f}_k(\hat{x}) e^{-j\theta k}$. Note that $\{\hat{f}_\theta\} : [0, 2\pi) \rightarrow L^2[0, X]$. In other words, $\{\hat{f}_\theta\}$ is an $L^2[0, X]$ -valued function defined on the unit circle, i.e. $\{\hat{f}_\theta\} \in L^2_{L^2[0, X]}[0, 2\pi)$. The

commutative diagram below schematically illustrates combined action of the lifting and z -transform on our system:

$$\begin{array}{ccc} L^2_{L^2[0, X]}[0, 2\pi] & \xrightarrow{\hat{G}} & L^2_{L^2[0, X]}[0, 2\pi] \\ z^{-1}|_{z=e^{j\theta}} \downarrow & & \uparrow z|_{z=e^{j\theta}} \\ L^2_{L^2[0, X]} & \xrightarrow{\bar{G}} & L^2_{L^2[0, X]} \\ W_X^{-1} \downarrow & & \uparrow W_X \\ L^2(-\infty, \infty) & \xrightarrow{G} & L^2(-\infty, \infty) \end{array}$$

3 Representation of the Lifted System

In this section, we first develop a convenient representation of the transforms of lifted signals, and then use it to obtain a representation of spatially periodic systems. In order to ‘visualize’ the transform $\hat{f}_\theta(\hat{x})$ in terms of the original function, we have to find an appropriate way to ‘view’ $\hat{f}_\theta(\hat{x})$ for each fixed $\theta \in [0, 2\pi)$. Since for every pre-specified θ , \hat{f}_θ is a function (in \hat{x}) over $[0, X]$, it is desirable to find a basis of $L^2[0, X]$ in which the representation of \hat{f}_θ is easily determined. The following Lemma from [2] shows that the θ -parameterized basis functions of the form (3), for any given θ , represent an orthonormal basis of $L^2[0, X]$. It turns out that this basis is very suitable for achieving our objective.

Lemma 1 *The θ -parameterized set of functions $\{\vartheta_{\theta, n}\}_{n \in \mathbb{Z}}$ of the form*

$$\vartheta_{\theta, n}(\hat{x}) := \frac{1}{\sqrt{X}} e^{j \frac{2\pi n + \theta}{X} \hat{x}}, \quad (3)$$

for each $\theta \in [0, 2\pi)$ represents a complete orthonormal basis of $L^2[0, X]$.

The following Proposition shows that the coefficients in the $\vartheta_{\theta, n}(\hat{x})$ -basis expansion of $\hat{f}_\theta(\hat{x})$ can be determined from the Fourier transform of the original function $f(x)$.

Proposition 2 *Let $f(x) \in L^2(-\infty, \infty)$ have a Fourier transform $F(k_x)$*

$$F(k_x) := \int_{-\infty}^{\infty} f(x) e^{-jk_x x} dx. \quad (4)$$

Then $\hat{f}_\theta(\hat{x})$ can be represented as

$$\hat{f}_\theta(\hat{x}) = \frac{1}{\sqrt{X}} \sum_{n \in \mathbb{Z}} F\left(\frac{2\pi n + \theta}{X}\right) \vartheta_{\theta, n}(\hat{x}). \quad (5)$$

We now apply these representations to find the corresponding representations of the underlying operators in the $\vartheta_{\theta, n}(\hat{x})$ -basis.

3.1 Spatially-Invariant Operators

Proposition 2 enables us to find the spectral coefficients in the $\vartheta_{\theta,n}(\hat{x})$ -basis expansion of $\hat{f}_{\theta}(\hat{x})$ as ‘samples’ of the Fourier transform $F(k_x)$. This practically means that for a spatially invariant PDE [3] of the form

$$\partial_t \psi(x, t) = \mathcal{A}(\partial_x) \psi(x, t), \quad (6)$$

whose frequency domain equivalent is given by

$$\dot{\Psi}(k_x, t) = \mathcal{A}(jk_x) \Psi(k_x, t),$$

the representation of the transformed lifted system

$$\partial_t \hat{\psi}_{\theta}(\hat{x}, t) = \hat{\mathcal{A}}_{\theta} \hat{\psi}_{\theta}(\hat{x}, t),$$

in the basis $\{\vartheta_{\theta,n}(\hat{x})\}_{n \in \mathbb{Z}}$, is completely determined by

$$\dot{\Psi}_{\theta}(t) = A_{\theta} \Psi_{\theta}(t),$$

where $\Psi_{\theta}(t) := [\dots \Psi(\frac{\theta}{X}, t) \Psi(\frac{2\pi+\theta}{X}, t) \dots]^T$, and $A_{\theta} := \text{diag}\{\mathcal{A}(j\frac{2\pi n+\theta}{X})\}_{n \in \mathbb{Z}}$. In other words, the matrix representation of $\hat{\mathcal{A}}_{\theta}$ in the chosen basis is simply $\text{diag}\{\mathcal{A}(j\frac{2\pi n+\theta}{X})\}_{n \in \mathbb{Z}}$, where $\mathcal{A}(\cdot)$ is the symbol used for the original PDE operator. For example, in the case of a heat equation on $L^2[-\infty, \infty]$, $\partial_t \psi(x, t) = \partial_x^2 \psi(x, t)$, the properties of our system are fully determined by $\text{diag}\{\dots, -(\frac{-2\pi+\theta}{X})^2, -(\frac{\theta}{X})^2, -(\frac{2\pi+\theta}{X})^2, \dots\}$, where parameter θ takes all values between 0 and 2π .

3.2 Periodic Gains

To apply this theory to an arbitrary PDE with periodic coefficients we need a method for lifting the part of the system which contains the periodic term. Suppose that we have a problem of the form (6) where the operator \mathcal{A} can be represented as $\mathcal{A}(\partial_x) := \mathcal{A}_1(\partial_x) + f(x)$, with \mathcal{A}_1 a spatially invariant operator and $f(x)$ an X -periodic function, $f(x) = f(x + X)$. Since we know how to lift the spatially invariant part of our system, we now illustrate a way for lifting a periodic gain. Let

$$\phi(x, t) := f(x) \psi(x, t). \quad (7)$$

Due to X -periodicity of the function f , $\bar{f}_k(\hat{x}) := f(kX + \hat{x}) = f(\hat{x})$, $\forall k \in \mathbb{Z}$, the lifted signal is given by

$$\bar{\phi}_k(\hat{x}, t) := f(\hat{x}) \bar{\psi}_k(\hat{x}, t), \quad k \in \mathbb{Z}, \quad 0 \leq \hat{x} \leq X. \quad (8)$$

After applying the z -transform and evaluating it on the unit circle, (8) simplifies to

$$\hat{\phi}_{\theta}(\hat{x}, t) = f(\hat{x}) \hat{\psi}_{\theta}(\hat{x}, t), \quad \theta \in [0, 2\pi), \quad 0 \leq \hat{x} \leq X,$$

which practically means that the lifted operator is just multiplication by $f(\hat{x})$ and is constant in θ . Our objective is to find the representation of this operator in the $\{\vartheta_{\theta,n}(\hat{x})\}_{n \in \mathbb{Z}}$ -basis. Since $f(x)$ is a periodic function, it can be decomposed into its Fourier series, $f(x) = \sum_{m \in \mathbb{Z}} a_m e^{j\frac{2\pi m}{X}x}$. By applying the Fourier transform on (7) we obtain

$$\begin{aligned} \Phi(k_x, t) &= \int_{-\infty}^{\infty} f(x) \psi(x, t) e^{-jk_x x} dx \\ &= \sum_{m \in \mathbb{Z}} a_m \int_{-\infty}^{\infty} \psi(x, t) e^{-j(k_x - \frac{2\pi m}{X})x} dx \\ &= \sum_{m \in \mathbb{Z}} a_m \Psi(k_x - \frac{2\pi m}{X}, t). \end{aligned} \quad (9)$$

Using Proposition 2, we express $\hat{\phi}_{\theta}(\hat{x}, t)$ as

$$\begin{aligned} \hat{\phi}_{\theta}(\hat{x}, t) &= \frac{1}{\sqrt{X}} \sum_{n \in \mathbb{Z}} \Phi(\frac{2\pi n + \theta}{X}, t) \vartheta_{\theta,n}(\hat{x}) \\ &= \frac{1}{\sqrt{X}} \sum_{n,m \in \mathbb{Z}} a_m \Psi(\frac{2\pi(n-m)+\theta}{X}, t) \vartheta_{\theta,n}(\hat{x}), \end{aligned}$$

where, based on (9), $\Phi(\frac{2\pi n + \theta}{X}, t)$ is determined by

$$\begin{aligned} \Phi(\frac{2\pi n + \theta}{X}, t) &= \sum_{m \in \mathbb{Z}} a_m \Psi(\frac{2\pi(n-m)+\theta}{X}, t) \\ &= \sum_{k \in \mathbb{Z}} a_{n-k} \Psi(\frac{2\pi k + \theta}{X}, t), \end{aligned}$$

or equivalently in the matrix form $\Phi_{\theta}(t) = \Gamma \Psi_{\theta}(t)$, where Γ is a bi-infinite Toeplitz matrix defined by

$$\Gamma := \begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ & \ddots & a_0 & a_{-1} & a_{-2} & & \\ & & \ddots & a_1 & a_0 & a_{-1} & \ddots \\ & & & a_2 & a_1 & a_0 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}. \quad (10)$$

Thus, we conclude that the matrix representation of a spatial-periodic gain in the $\{\vartheta_{\theta,n}(\hat{x})\}_{n \in \mathbb{Z}}$ -basis is completely determined by a Toeplitz matrix whose elements correspond to the Fourier coefficients

$$a_m := \frac{1}{X} \int_0^X f(x) e^{-j\frac{2\pi m}{X}x} dx.$$

For example,

$$f(x) = \varepsilon \sin(\frac{2\pi}{X}x) \Rightarrow a_m = \begin{cases} j\frac{\varepsilon}{2} & m = -1, \\ -j\frac{\varepsilon}{2} & m = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f(x) = \varepsilon \cos(\frac{2\pi}{X}x) \Rightarrow a_m = \begin{cases} \frac{\varepsilon}{2} & m = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Feedback Interconnections

The procedure described in § 3.1 and § 3.2 can be applied to obtain a representation of a system with spatially periodic coefficients on $L^2(-\infty, \infty)$ of the form

$$\begin{aligned} \partial_t \psi(x, t) &= (\mathcal{A}_1 + \mathcal{B}_1 f \mathcal{C}) \psi(x, t) + \mathcal{B}_2 u(x, t) \\ &=: \mathcal{A} \psi(x, t) + \mathcal{B} u(x, t), \end{aligned} \quad (11a)$$

$$y(x, t) = \mathcal{C} \psi(x, t), \quad (11b)$$

where \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{B}_2 , and \mathcal{C} are spatially-invariant operators, f is a given X -periodic matrix valued function, u is a distributed input, y is a distributed output, and ψ is a distributed state. We can equivalently represent system (11) as shown in Figure 1.

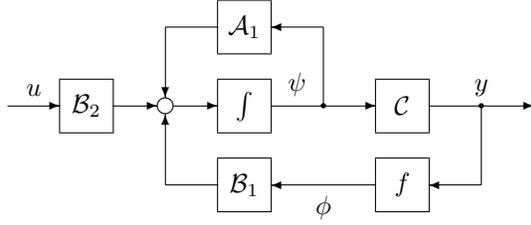


Figure 1: Block diagram of system (11) with spatially invariant operators \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{B}_2 , and \mathcal{C} , and an X -periodic matrix valued function f .

Since the lifting is invariant under feedback [1] the transformed lifted system can be rewritten as

$$\begin{aligned} \partial_t \hat{\psi}_\theta(\hat{x}, t) &= (\hat{\mathcal{A}}_{1\theta} + \hat{\mathcal{B}}_{1\theta} f(\hat{x}) \hat{\mathcal{C}}_\theta) \hat{\psi}_\theta(\hat{x}, t) + \hat{\mathcal{B}}_{2\theta} \hat{u}_\theta(\hat{x}, t) \\ &=: \hat{\mathcal{A}}_\theta \hat{\psi}_\theta(\hat{x}, t) + \hat{\mathcal{B}}_\theta \hat{u}_\theta(\hat{x}, t), \end{aligned} \quad (12a)$$

$$\hat{y}_\theta(\hat{x}, t) = \hat{\mathcal{C}}_\theta \hat{\psi}_\theta(\hat{x}, t), \quad (12b)$$

where $\{\hat{\mathcal{A}}_{1\theta}, \hat{\mathcal{B}}_{1\theta}, \hat{\mathcal{B}}_{2\theta}, \hat{\mathcal{C}}_\theta\}$ and $f(\hat{x})$ can be lifted according to the procedure described in § 3.1 and § 3.2, respectively. Thus, in the $\{\vartheta_{\theta, n}(\hat{x})\}_{n \in \mathbb{Z}}$ -basis system (12) is represented by

$$\begin{aligned} \dot{\Psi}_\theta(t) &= (A_{1\theta} + B_{1\theta} \Xi C_\theta) \Psi_\theta(t) + B_{2\theta} U_\theta(t) \\ &=: A_\theta \Psi_\theta(t) + B_\theta U_\theta(t), \end{aligned} \quad (13a)$$

$$Y_\theta(t) = C_\theta \Psi_\theta(t), \quad (13b)$$

where, for example, $A_{1\theta}$ is a bi-infinite block diagonal matrix determined by $\text{diag}\{\mathcal{A}_1(j \frac{2\pi n + \theta}{X})\}_{n \in \mathbb{Z}}$, and Ξ is a block Toeplitz matrix whose elements can be determined based on (10).

As an example, consider $\partial_t \psi = \partial_x^2 \psi + \cos(\frac{2\pi}{X} x) \partial_x^2 \psi + \partial_x u$. In this case, the operators on the right-hand side of (13a) become $A_\theta = (I + \Gamma) \text{diag}\{-\frac{2\pi n + \theta}{X}\}_{n \in \mathbb{Z}}$ and $B_\theta = \text{diag}\{j \frac{2\pi n + \theta}{X}\}_{n \in \mathbb{Z}}$, respectively.

We emphasize that systems (11) and (13) have same stability properties. In particular, it can be shown that the spectrum of the generator of (11) is determined by $\sigma\{\mathcal{A}\} = \bigcup_{\theta \in [0, 2\pi)} \sigma\{A_\theta\}$. Furthermore, since both W_X and W_X^{-1} represent bijective linear isometries it follows that the lifting preserves system norms [1]. To illustrate this, we apply the temporal Fourier transform on (11) and (13) to obtain $y(x, \omega) = [\mathcal{C}(j\omega - \mathcal{A})^{-1} \mathcal{B}u(\omega)](x) =: [\mathcal{H}(\omega)u(\omega)](x)$, and $Y_\theta(\omega) = C_\theta(j\omega - A_\theta)^{-1} B_\theta U_\theta(\omega) =: H_\theta(\omega)U_\theta(\omega)$, respectively. We remark that for any given ω , $\mathcal{H}(\omega)$ represents a spatial operator that maps $u(\omega)$ into $\psi(\omega)$. On the other hand, $H_\theta(\omega)$ is a multiplication operator parameterized by two frequencies: $\omega \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Properties of these two operators are closely related. For example, it can be shown that the \mathcal{H}_∞ and \mathcal{H}_2

norms of operator \mathcal{H} can be determined as

$$\begin{aligned} \|\mathcal{H}\|_\infty &:= \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{\mathcal{H}(\omega)\} \\ &= \sup_{\theta \in [0, 2\pi)} \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{H_\theta(\omega)\}, \\ \|\mathcal{H}\|_2^2 &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{H}(\omega)\|_{HS}^2 d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \|H_\theta(\omega)\|_{HS}^2 d\omega d\theta, \end{aligned}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt operator norm. It is noteworthy that the \mathcal{H}_2 norm of stable systems can be determined based on solutions of the operator Lyapunov equations of the form

$$\begin{aligned} \mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* &= -\mathcal{B}\mathcal{B}^*, \\ \mathcal{A}^*\mathcal{Q} + \mathcal{Q}\mathcal{A} &= -\mathcal{C}^*\mathcal{C}, \end{aligned}$$

as $\|\mathcal{H}\|_2^2 = \text{trace}\{\mathcal{P}\mathcal{C}^*\mathcal{C}\} = \text{trace}\{\mathcal{Q}\mathcal{B}\mathcal{B}^*\}$, where \mathcal{A}^* , \mathcal{B}^* , and \mathcal{C}^* represent adjoints of operators \mathcal{A} , \mathcal{B} , and \mathcal{C} , respectively. In general it is very difficult to solve operator Lyapunov equations for \mathcal{P} and \mathcal{Q} . However, it is much easier to work with the lifted operators and compute the \mathcal{H}_2 norm of stable system (11) as

$$\begin{aligned} \|\mathcal{H}\|_2^2 &= \int_0^{2\pi} \text{trace}\{P_\theta C_\theta^* C_\theta\} d\theta \\ &= \int_0^{2\pi} \text{trace}\{Q_\theta B_\theta B_\theta^*\} d\theta, \end{aligned}$$

with P_θ and Q_θ being the solutions of

$$\begin{aligned} A_\theta P_\theta + P_\theta A_\theta^* &= -B_\theta B_\theta^*, \\ A_\theta^* Q_\theta + Q_\theta A_\theta &= -C_\theta^* C_\theta. \end{aligned}$$

Results of this section are summarized in the following theorem.

Theorem 3 *The feedback stability and the system norms are preserved under the lifting. In particular, the spectrum of the generator of (11) is equal to the spectrum of the generator of (13) as θ ranges over all possible values, that is*

$$\sigma\{\mathcal{A}\} = \bigcup_{\theta \in [0, 2\pi)} \sigma\{A_\theta\}.$$

Furthermore, the \mathcal{H}_∞ and \mathcal{H}_2 norms of operator mapping input u into output y are determined by

$$\begin{aligned} \|\mathcal{H}\|_\infty &:= \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{\mathcal{H}(\omega)\} \\ &= \sup_{\theta \in [0, 2\pi)} \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{H_\theta(\omega)\}, \\ \|\mathcal{H}\|_2^2 &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{H}(\omega)\|_{HS}^2 d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \|H_\theta(\omega)\|_{HS}^2 d\omega d\theta. \end{aligned}$$

4 Examples of PDEs with Periodic Coefficients

In this section we illustrate with two examples how stability properties of PDEs can be changed by introducing feedback terms with spatially periodic coefficients.

Our results reveal that, depending on the values of amplitude and frequency of a periodic gain, the original equation can be either stabilized or destabilized. This phenomenon is referred to as ‘parametric resonance’ and is encountered in many problems of physical significance. Furthermore, we show how our results can lead to an *analytic expression* for the eigenvalues of the underlying operators, for a special class of PDEs with periodic coefficients. This is to some degree a surprising discovery since our original intention had been to develop a tool for a *numerical approximation* of transformed lifted operators.

First example

In [4], the periodic solutions of the Ginzburg-Landau (GL) equation are studied. This equation results from nonlinear stability theory and appears in the analysis of many relevant fluid mechanics problems: the Bénard problem, the Taylor problem, Tollmien-Schlichting waves, gravity waves, etc. (for more details see [4] and references therein). It is worth mentioning that the GL equation describes the evolution of a slowly varying complex amplitude of a neutral plane wave.

The linearization of this equation around its limit cycle solution $\phi_0 := f(x) \exp(j\Omega t)$, $f(x) = f(x + X)$, results in a PDE with periodic coefficients of the form

$$\partial_t \phi = z_1 \phi + z_2 \partial_x^2 \phi + z_3 (2|f|^2 \phi + f^2 \phi^*), \quad (14)$$

where ϕ^* is the complex-conjugate of a field ϕ , and $z_1 := \rho - j\Omega$, $z_2 := c_0 + j$, $z_3 := j - \rho$, with c_0 , Ω , and ρ being real valued parameters. Furthermore, $\rho := c_0/c_1$, $0 \leq c_0^2 \leq c_1$.

If we introduce the following notation

$$\psi := \begin{bmatrix} \phi \\ \phi^* \end{bmatrix}, \quad (15)$$

we can rewrite (14) as

$$\partial_t \psi := \mathcal{A}(\partial_x) \psi, \quad (16)$$

where the operator \mathcal{A} is given by

$$\mathcal{A} := \begin{bmatrix} z_1 + z_2 \partial_x^2 + 2z_3 |f|^2 & z_3 f^2 \\ (z_3 f^2)^* & z_1^* + z_2^* \partial_x^2 + 2z_3^* |f|^2 \end{bmatrix}.$$

Clearly, \mathcal{A} can be separated into a sum of a spatially invariant operator and a matrix-valued periodic function, which lends itself to the application of the results derived in § 3. We use these results for a numerical approximation of \mathcal{A} and investigate its eigenvalues. In particular, we assume $f(x) := \varepsilon \cos(\frac{2\pi}{X}x)$ and consider a problem of the stability of system (14). We note that in this case (14) cannot be interpreted as the linearized GL equation since ϕ_0 , for the above given $f(x)$, is no longer a limit cycle solution of the nonlinear GL equation. Nevertheless, this problem is worth investigating because it represents an example of a system whose stability can be changed by feedback terms with spatially periodic gains.

Figure 2 illustrates the maximal real part of the operator \mathcal{A} eigenvalues, $\max(\operatorname{Re}\{\lambda(\mathcal{A})\})$, as a function of ε and X ,

for $c_0 = 0.4$, $\rho = 0.4$, and $\Omega = 5$. It can be readily shown that for $\varepsilon = 0$, $\max(\operatorname{Re}\{\lambda(\mathcal{A})\}) = \rho = 0.4$, which means that system (14) is ‘open-loop’ unstable. Clearly, ‘closing a loop’, by introducing the spatially periodic gains in feedback, changes the value of $\max(\operatorname{Re}\{\lambda(\mathcal{A})\})$ significantly, as illustrated in Figure 2.

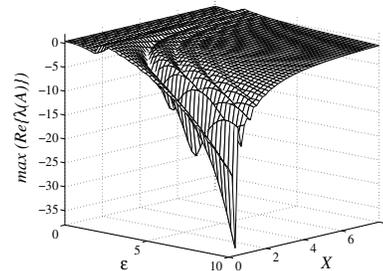


Figure 2: $\max(\operatorname{Re}\{\lambda(\mathcal{A})\})$ as a function of ε and X for $f(x) := \varepsilon \cos(\frac{2\pi}{X}x)$, $c_0 = 0.4$, $\rho = 0.4$, and $\Omega = 5$.

Figure 3 shows $\max(\operatorname{Re}\{\lambda(\mathcal{A})\})$ as a function of ε for $X \approx 1.31$. This plot further illustrates the previously mentioned changes in $\max(\operatorname{Re}\{\lambda(\mathcal{A})\})$, and reveals an interesting feature of this example. Namely, the regions of instability and stability repeat in an alternating arrangement for certain values of the amplitude ε .

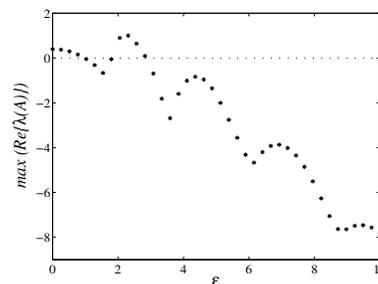


Figure 3: $\max(\operatorname{Re}\{\lambda(\mathcal{A})\})$ as a function of ε for $f(x) := \varepsilon \cos(\frac{2\pi}{X}x)$, $X \approx 1.31$, $c_0 = 0.4$, $\rho = 0.4$, and $\Omega = 5$.

Second example

In [5], a class of nonlinear PDEs with periodic plane wave solutions including forms of the Schrödinger and generalized KdV equations is considered. The linearization of these equations around a plane wave yields the following PDE with periodic coefficients:

$$\partial_t \phi(x, t) = p(\partial_x) \phi(x, t) + \alpha e^{2j\frac{2\pi}{X}x} \phi^*(x, t), \quad (17)$$

where p is a polynomial with complex coefficients, and α is a complex scalar. We can define the state by (15), which transforms (17) into (16), with

$$\mathcal{A} := \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} := \begin{bmatrix} p(\partial_x) & \alpha e^{2j(\frac{2\pi}{X})x} \\ \alpha^* e^{-2j(\frac{2\pi}{X})x} & p^*(\partial_x) \end{bmatrix}.$$

Operator \mathcal{A} can be separated into a sum of a spatially invariant operator and a matrix-valued periodic function, and the previously described procedure transforms (16) into

$$\dot{\Psi}_\theta(t) = A_\theta \dot{\Psi}_\theta(t). \quad (18)$$

Bi-infinite Toeplitz matrices $A_{\theta 12}$ and $A_{\theta 21}$ are defined in (10) with the elements of $A_{\theta 12}$ given by

$$a_m = \begin{cases} \alpha & m = 2, \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

and the elements of $A_{\theta 21}$ given by

$$a_m = \begin{cases} \alpha^* & m = -2, \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Furthermore, if we define $p_n(\theta) := p(j\frac{2\pi n + \theta}{X})$, then

$$A_{\theta 11} = \text{diag}\{p_n(\theta)\}_{n \in \mathbb{Z}}, \quad A_{\theta 22} = \text{diag}\{p_n^*(\theta)\}_{n \in \mathbb{Z}}.$$

The eigenvalues of A_θ are computed by defining $G_\theta := \lambda I - A_\theta$ and using the formula

$$\det(G_\theta) = \det(G_{\theta 11}) \det(G_{\theta 22} - G_{\theta 21} G_{\theta 11}^{-1} G_{\theta 12}).$$

Noting that $G_{\theta 12}$ and $G_{\theta 21}$ are just scaled backward and forward shift operators, respectively, then $\det(\lambda I - A_\theta)$ is computed to be

$$\prod_{n \in \mathbb{Z}} (\lambda - p_n(\theta)) \prod_{n \in \mathbb{Z}} (\lambda - p_n^*(\theta) - \frac{|\alpha|^2}{\lambda - p_{n-2}(\theta)}) = \prod_{n \in \mathbb{Z}} \{(\lambda - p_{n-2}(\theta))(\lambda - p_n^*(\theta)) - |\alpha|^2\},$$

which yields the eigenvalues $\lambda_n(\theta)$ of A_θ equal to

$$\frac{1}{2} \{p_{n-2}(\theta) + p_n^*(\theta) \pm \sqrt{4|\alpha|^2 + (p_{n-2}(\theta) - p_n^*(\theta))^2}\}.$$

This result can be applied to the nonlinear Schrödinger equation

$$j \partial_t \varphi(z, t) = - \partial_z^2 \varphi(z, t) - V(\varphi \varphi^*) \varphi(z, t), \quad (21)$$

with potential $V(\phi \phi^*)$ that satisfies the dispersion equation $V(a^2) = (\frac{2\pi}{X})^2 - (\frac{2\pi}{T})^2$. As illustrated in [5], (21) exhibits a periodic plane wave solution $ae^{j(\frac{2\pi}{X}z - \frac{2\pi}{T}t)}$. The linearization of (21) around this solution, together with a coordinate transformation of the form $x := z - \frac{X}{T}t$, yields (17) with

$$\begin{aligned} p(\partial_x) &:= \frac{X}{T} \partial_x + j \{ \partial_x^2 + V(a^2) + a^2 V'(a^2) \}, \\ \alpha &:= ja^2 V'(a^2). \end{aligned}$$

It is readily shown, using the previously derived formula for $\lambda_n(\theta)$, that the generator of (17) has purely imaginary eigenvalues if and only if $V'(a^2) \leq 0$. On the other hand, if $V'(a^2) > 0$ then there exist an eigenvalue of \mathcal{A} with a positive real part. The same conclusion has been derived in [5] using the technique which is well-suited for systems that can be represented by (17). It is worth noting that the spatio-temporal lifting can be used for analysis of stability and input-output norms of general linear PDEs with periodic coefficients on $L^2(-\infty, \infty)$.

5 Concluding Remarks

We have developed the appropriate framework for analysis of stability properties and system norms of distributed parameter systems with spatially periodic coefficients. It has been shown that the main ideas of a well known temporal lifting technique for linear periodic ODEs are readily extendable to PDEs in which some of the coefficients are spatially periodic functions. A particular basis set has been used to obtain convenient representations of transformed lifted signals and spatially periodic systems. It has been also illustrated, by means of two examples, how stability properties of spatially invariant PDEs can be changed when a spatially periodic feedback is introduced.

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