UNDERSTANDING MECHANICAL DOMAIN PARAMETRIC RESONANCE IN MICROCANTILEVERS

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***Abstract*— In this paper we present a mathematical model for the dynamics of an electrostatically actuated micro-cantilever. For the common case of cantilevers excited by a periodic voltage, we show that the underlying linearized dynamics are those of a periodic system described by a Mathieu equation. We present experimental results that confirm the validity of the model, and in particular illustrate that parametric resonance phenomena occur in capacitively actuated micro-cantilevers. The combined parametric/harmonic mode of operation is investigated as well and experimental data are provided.**

I. Introduction

Resonant mode operation of micro and nano-scale oscillators have gained wide interest for applications including filters, amplifiers, non-linear mixers, atomic scale imaging, biological and chemical sensors.

The working principle of these devices is based on measurement of displacement. Hence, their sensitivity strongly depends on the smallest detectable motion. As the size of the components gets smaller, so does the magnitude of their displacement, with the result that the transduction mechanism has to operate close to the background noise. This drawback motivates the interest in amplification schemes in the mechanical domain, including *parametrically excited* excited systems, which are systems described by differential equations in which the input appears as a time-dependent coefficient. In particular, their characteristic is that large responses can be generated even if the excitation frequency is far away from the system's natural frequency.

The device that we propose is an electrostatically actuated microcantilever. Cantilever geometries are particularly interesting, due to their wide range of applications, including small force detection [1], [2] , AFM, mechanical filters for telecommunication [3], and chemical sensor arrays [4]. In our design the microcantilever constitutes the movable plate of a capacitor and its displacement is controlled by the voltage applied across the plates.

In this paper, we present a model for this electrostatically actuated microcantilever. Using simple parallel plate theory and for the common case of sinusoidal forcing, the dynamics are governed by the Mathieu equation. We produce experimental evidence that validates the mathematical model, including a mapping of the first instability region of the Mathieu equation. While parametric amplification has been discussed in small scale resonant systems [5], [6], [7], [8], [9], [1], to our knowledge this is the first experimental mapping of the first parametric instability region in a micro-cantilever resonator.

The paper is organized as follows: In Section 2 we develop the mathematical model of an electrostatically actuated cantilever. In Section 3 we present the experimental results that validate the model including, in particular, the mapping of the first instability region of the Mathieu equation. In Section 4 we analyze the behavior of the cantilever in combined harmonic and parametric resonance mode of operation. Finally, we present our conclusions in Section 5.

II. Model Description For a Micro Capacitive Cantilever

From parallel plate theory, neglecting the asymmetry in electric field for a cantilever, the attractive force, F_a, between the capacitor plates generated by applying a voltage $V(t)$, can be expressed as

$$F_a = \frac{1}{2}\frac{\epsilon_o A}{d^2}\frac{V^2(t)}{(1-\frac{z}{d})^2} \approx \frac{1}{2}\frac{\epsilon_o A}{d^2}(1+2\frac{z}{d})V^2(t),$$

where the approximation holds when $\frac{z}{d} << 1$. Here ϵ_o is the permittivity in vacuum, A is the area of the plates, d is the gap between them and z is the vertical displacement of the cantilever from its rest position.

Only few algebraic steps are necessary to derive the equation of motion of the cantilever, which if $V(t) = V_o \cos\omega_o t$, is given by

$$z'' + cz' + (a - 2q\,\cos 2t)z = u_f(t), \qquad (1)$$

where the prime denotes the derivative with respect to the scaled time $\tau = \omega_o t$; c is a small damping coefficient, presumed viscous, $a = \frac{k}{m\omega_o^2} - \frac{1}{2}\frac{\epsilon_o A V_o^2}{md^3\omega_o^2}$, k is the spring constant of the cantilever, $q = \frac{\epsilon_o A V_o^2}{4md^3\omega_o^2}$, and $u_f(t) = q\,d\cos^2(t)$.

Equation (1) is an instance of the well-known Mathieu equation. In the next section we will briefly discuss its peculiar stability properties, and demonstrate how they can be advantageously exploited from an engineering point of view.

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III. Experimental Validation of the Cantilever Model

The device we have used in our experimental setup was a $200\mu m \times 50\mu m \times 2\mu m$ highly doped polysilicon cantilever, fabricated using the MUMPS/CRONOS process, and with a gap between the electrodes of about $2\mu m$. The mechanical response of the cantilever was tested in vacuum ($p = 8mT$), using laser vibrometry [11] to measure its displacement and velocity near the free end, when electrostatically driven with different AC voltage signals. Measurement of the frequency response, subject to small excitation, as in [11] yields a first natural frequency of approximately $f_r = 50800Hz$, a damping coefficient $c = 2.1 \times 10^{-4}$, and a quality factor of $Q = 2200$. The values of these parameters were confirmed by time domain identification experiments as well.

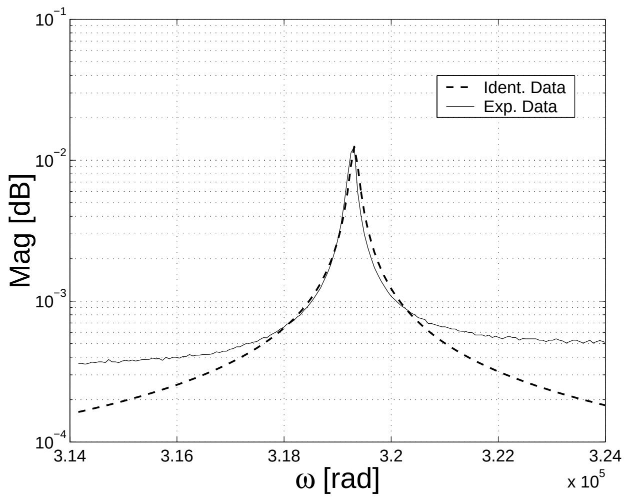

Fig. 1. Frequency response of the capacitive cantilever: the dashed line corresponds to measured data, the solid one is its least square fit.

As the amplitude of the driving signal increases, so does the value of q and this linear time-invariant approximation of the model is no longer appropriate. Therefore, we have to return to the original equation (1).

A. The Mathieu equation and Parametric Resonance

Extensive literature exists on the standard form of the Mathieu equation,

$$z'' + (a - 2q\,\cos 2t)z = 0. \tag{2}$$

Its stability properties have been thoroughly investigated as a function of a and q. By means of perturbation analysis methods, it is possible to determine the values of these parameters that correspond to unstable behavior.

In particular, it is not difficult to prove that instabilities occur at $a = n^2$, $n \in \mathbb{N}$ [15], and that the boundaries of the first instability region, for small values of q, are given by $a = 1 \pm 2q$. In terms of the physical parameters of the device, the driving frequencies that cause unstable responses in the system are given by

$$\omega_o \approx \frac{2\omega_r}{n} \qquad n \in \mathbb{N}\,,$$

while the boundaries of the first instability region, in terms of frequency and amplitude of excitation, are defined by

$$\omega_o^2 = 4\omega_r^2 - 4(1 \mp \frac{1}{2})\frac{\epsilon A V_o^2}{md^3}.$$

It is worth noting that the presence of a damping term, whose existence we have neglected so far, has the effect of shifting the tongues upwards in the a-q parameter space requiring a minimum threshold voltage for parametric resonance. This is the reason why parametric resonance is difficult to observe at the macroscale. The experimentally determined boundary of the first instability region is shown in Fig.2.

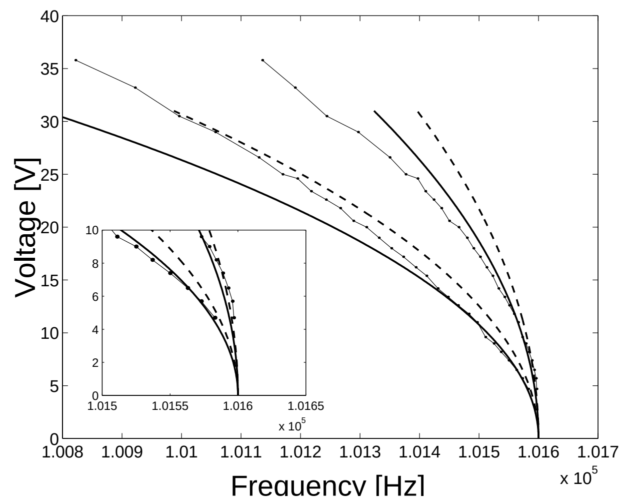

Fig. 2. The thin line (overlaying the dots) is an experimentally determined map of the first parametric resonance region in the microcantilever. The dotted line shows the analytical result, using the Q and natural frequency determined from experiment, and the solid line shows the analytical result when using Q and natural frequency and electrostatic stiffness determined from experiment.

Inside the instability region the cantilever oscillation does not grow unbounded, but rather saturates at a large amplitude. This is explained by nonlinear effects, which cause the system to settle into a steady state response [10]. For large oscillation amplitudes, both the linear spring model and the electrostatic force need to be corrected by adding cubic terms [13]. Hence the equation of motion (1) becomes

$$z'' + (a - 2q\,\cos 2t)z + a_2 z^3 = u_f(t), \tag{3}$$

where a_2 denotes the effective cubic stiffness of the beam, which includes both electrostatic and structural contributions. What we observe when driving the cantilever in parametric resonance regime is a subharmonic 2:1 oscillation of the beam [10], which vibrates at half the frequency of excitation, as shown in Fig.3 a)). Note also that during the transition from non-parametric to parametric region, the response shows a characteristic exponential growth (see Fig.3 b)).

Above the critical driving voltage amplitude, and for driving frequencies near the first parametric resonance, the response of the cantilever has the behavior depicted in Fig.4. The two curves represent data collected by sweeping the frequency from low to high (black '+' points) and from high to low

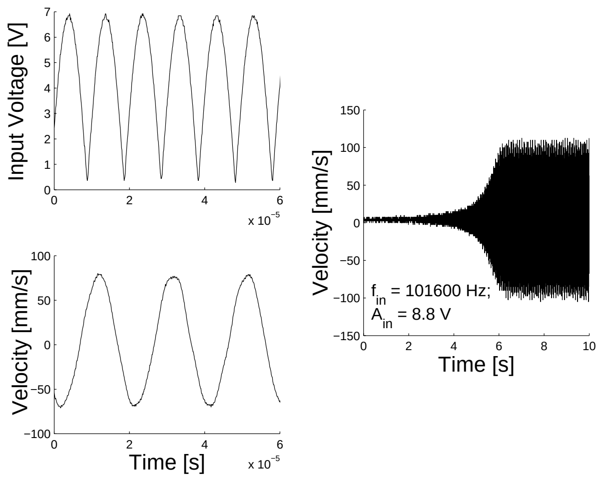

Fig. 3. Time series of a typical input and output inside the first 'tongue'. Note the response is at half the driving frequency.

(grey 'o' points), as indicated by the arrows. In area III,

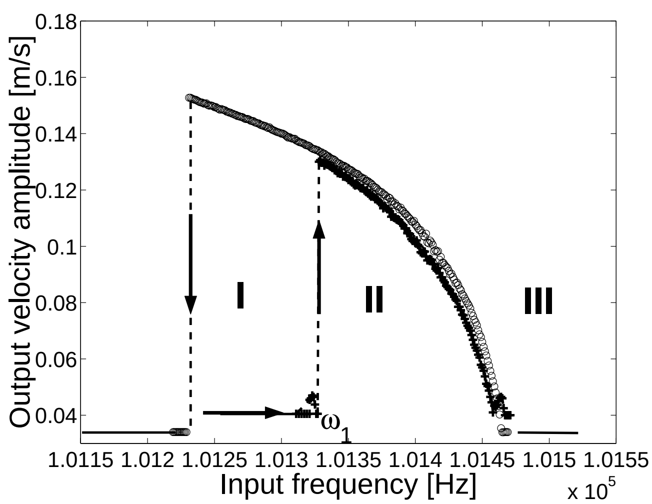

Fig. 4. Frequency response above critical driving voltage amplitude when operating near the first 'tongue' ($V_o = 10V$).

the system has one stable solution, which upon entering the parametric region II, bifurcates into a stable periodic motion and an unstable (experimentally unobservable) one. The periodic solution grows in amplitude as ω continues to decrease. When the system leaves the parametric region, area I, the stable periodic solution at some point collapses to the trivial solution, which is stable again.

The sharp transition of the output response (vertical segment of '$+$' data) marks the entrance into the parametric region. Since this transition always occurs for the same value ω_1, related to the resonant frequency of the beam, the phenomenon has potentially many applications, from the realization of mechanical filters to extremely sensitive mass sensors. The authors are pursuing both of these applications [13][14]

IV. Combined harmonic-parametric response with cubic nonlinearity

When driving the system with a sinusoidal voltage, $V(t) = V_{DC} + V_o \cos \omega t$, equation (3) becomes

$$z'' + cz' + (a - 2p\,\cos t - 2q\,\cos 2t)z + a_2 z^3 = u_f(t) + u_p(t), \tag{4}$$

where $p = \frac{\epsilon_o A V_{DC} V_o}{m d^3 w_o^2}$ and $u_p(t) = pd\cos t$. When the frequency of the driving signal is close to the resonant frequency w_o of the cantilever, the effect of the $\cos 2t$ term in the coefficient of z and the $\cos t$ term in the direct excitation are of the leading order. To obtain the first order parametric response, we can consider (4) to be a harmonic oscillator with a perturbation and re-write it as follows

$$z'' + cz' + z = \epsilon((-a_1 + 2q_1\,\cos 2t)z + a_{21}z^3 + p_1\cos t),$$

where $a = 1+\epsilon a_1$, $q_1 = q/\epsilon$, $a_{21} = a_2/\epsilon$, $p_1 = p/\epsilon$. By means of perturbation analysis methods, if we assume the solution to be of the form $z(t) = A(\eta)\cos t + B(\eta)\sin t$, with $\eta = \epsilon t$ corresponding to the slow varying time scale, we obtain the following slow flow equations,

$$\begin{aligned} \frac{dA}{d\eta} &= \frac{B}{2}(a_1 - 1) + \frac{3a_{21}B}{8}(A^2 + B^2) - p_1, \\ \frac{dB}{d\eta} &= -\frac{A}{2}(a_1 + 1) - \frac{3a_{21}A}{8}(A^2 + B^2). \end{aligned}$$

These non-linear coupled ODEs can be readily solved numerically to obtain the regions of parametric resonance. This analysis predicts bi-stable non-trivial response in certain regions in the driving frequencyamplitude plane. In fact, this has been observed experimentally and the steady-state frequency sweep response above a critical amplitude of drive is shown in Fig.5. The symbol '$\diamond$' corresponds to the frequency being

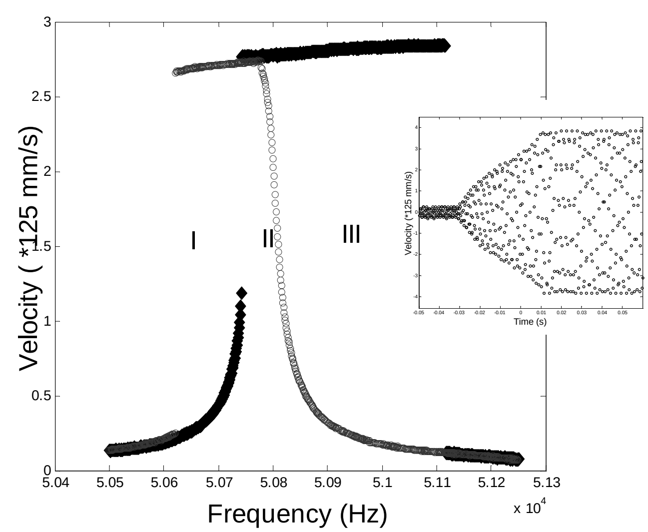

Fig. 5. Combined harmonic/parametric frequency response. Note that there are bistable regions (I and III) on either side of the parametric tongue (II).

swept up and the symbol 'o' to the frequency being swept down. The figure also shows the time response when the cantilever forcing is changed from a non-parametric region to a parametric region. Notice that the rate of growth, initially characteristic of harmonic resonance, becomes exponential, as expected for parametric mode of oscillation. Fig.6 shows the mapping of the parametric resonance region in the drive amplitude-frequency plane. The effect of the direct forcing

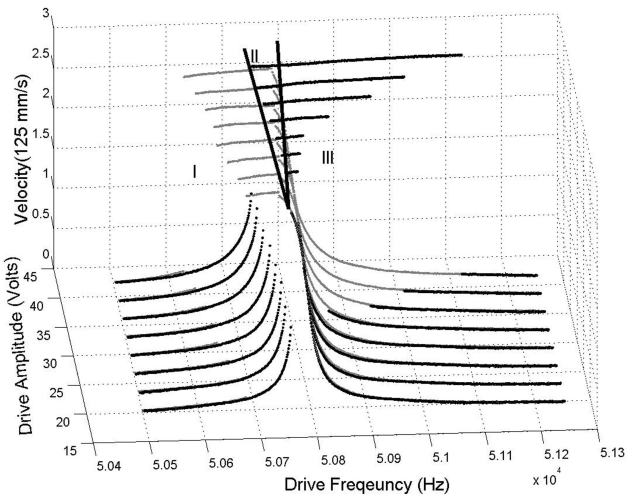

Fig. 6. The first instability region with combined harmonic and parametric excitation. Note that at the boundary between regions I and II, there is still a distinctly sharp transition between small and large amplitude response, making it useful for applications.

is to transform the trivial stable solution into a non-trivial frequency dependent stable periodic solution. When driven in region I or III, the response of the cantilever is one of the two stable states, depending on the initial conditions. This confirms what is seen in the experiments. The presence of the direct driving harmonic component changes the parametric frequency response of the cantilever [12]. The change can be explained if the direct harmonic component is smaller than the parametric drive by an order of magnitude (which is true in our case) by perturbation analysis of the governing equation. The overall response is richer than the pure parametric case because of the presence of bi-stable regions. The sharp transition from a small periodic response to a large one with change of drive frequency is also seen in this case. Hence applications that use these sharp transitions can be implemented in this mode as well. As it is extremely hard to avoid a harmonic forcing term in most sinusoidally driven applications, the presence of a sharp transition is extremely promising.

V. Conclusions

We present a thorough modeling, analysis and experimental verification of using mechanical domain nonlinearity to amplify motion in the resonant mode operation of a microscale cantilever. The model presented here takes into account nonlinearity in mechanical and electrostatic domains. We have provided experimental validation of the mathematical model, which included the mapping of the first region of instability of the Mathieu equation. We have analyzed the behavior of the oscillator when a DC offset term is added to the sinusoidal forcing signal and explored, the combined parametric/harmonic mode of operation. From this work, many sensing applications can be realized, utilizing the sharp transitions from non-resonant to resonant state, which are present in the combined as well as the isolated parametrically resonant state. Filters and sensors using this mechanism are being explored. In addition, an extension to multi-cantilever arrays is also being investigated. This result offers designers tangible guidelines needed to implement novel parametric devices.

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