$\mathcal{H}_2$-norm minimization for distributed continuous time systems: an input/output approach

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Abstract

In this paper we extend the result of Youla on the parametrization of output feedback stabilizing controllers to spatially-invariant distributed systems. We also introduce a new definition of $\mathcal{H}_2$ norm and solve the related optimization problem, using the parametrization obtained. Finally, we apply the methodology presented in the paper to the case of a multicantilever structure.

1 Introduction

Recent technological advances in the field of micro devices have made feasible the implementation of new control structures, consisting of large arrays of spatially distributed actuators and sensors. In turn, these architectures have generated an increasing interest in spatio-temporal systems, which represent the right mathematical tool to model and study the resulting systems.

An important subclass, within this large family, is represented by *spatially invariant* systems, whose dynamics are invariant with respect to translation in the spatial coordinates.

This property of spatial invariance turns out to be crucial. As a matter of fact, under this hypothesis, we can apply the Fourier transform in the spatial variable and associate with the distributed system a one dimensional parametric model, which is equivalent to the original one (see [1]).

We will now briefly introduce linear spatially-distributed continuous time systems. We refer the interested reader to [1, 2] for the main results concerning this topic.

In a state-space approach, spatially invariant distributed systems are described by a set of equations

$$\begin{aligned} \tfrac{d}{dt}x(t,k) &= A * x(t,k) + B * u(t,k) \\ y(t,k) &= C * x(t,k) + D * u(t,k) \end{aligned} \qquad (1)$$

where $t \in \mathbf{R}$ is the time variable, $k \in \mathbf{Z}$ is the spatial coordinate, A, B, C and D are linear convolution operators on $l_1(\mathbf{Z})$ and ' $*$ ' denotes the operation of spatial convolution.

If we apply the Fourier transform in the spatial domain, we can associate the two dimensional system (1) with an equivalent one dimensional parametric system

$$\begin{aligned} \dot{\hat{x}}(t,\theta) &= \hat{A}(\theta)\hat{x}(t,\theta) + \hat{B}(\theta)\hat{u}(t,\theta) \\ \hat{y}(t,\theta) &= \hat{C}(\theta)\hat{x}(t,\theta) + \hat{D}(\theta)\hat{u}(t,\theta). \end{aligned} \qquad (2)$$

The equivalence between (1) and (2) is established in [2, 1].

2 Controller parametrization

Given a plant P, the parametrization of all feedback stabilizing controllers is an important tool, that enables the development of synthesis techniques for robust control. As a matter of fact, once this parametrization is available, it is possible to set criteria to choose, among the admissible controllers, the one which better satisfies some other specific performance objective.

In the one dimensional case, the problem of parameterizing all stabilizing controllers has been solved by Youla using the coprime factorization technique.

The extension of this methodology to the class of spatially-invariant systems requires some care. First of all, we have to deal with the problem of the coprime factorization of a 2D transfer function.

While in the case of rational functions with real coefficients such a factorization always exists, in the 2D case the result is not immediate.

Lemma 2.1 *[3] Consider a transfer function $G(s,\theta)$ which has a realization (2). If $\forall\theta \in [0, 2\pi]$, $(A(\theta), B(\theta))$ is stabilizable and $(C(\theta), A(\theta))$ is detectable, then there*

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exist a right and a left coprime factorization of $G(s,\theta)$

$$G = N_r D_r^{-1} = D_l^{-1} N_l, \tag{3}$$

over the ring R *of rational functions in* s, *with coefficients in* $l_1(\mathbf{Z})$, $R = [l_1(\mathbf{Z})](s)$.

From this lemma, it follows that the parametrization of all stabilizing controllers in terms of their input/output transfer function is given by

$$\begin{aligned} C &= \{(Y_r - QN_l)^{-1}(-X_r + QD_l) \quad Q \in R\} \\ &= \{(-X_l + D_r Q)(Y_l + N_r Q)^{-1} \quad Q \in R\}. \end{aligned}$$

3 The $\mathcal{H}_2$ norm for two dimensional systems

One of the typical performance objectives in the one dimensional case is the minimization of the $\mathcal{H}_2$ norm of the system transfer matrix, which corresponds in the time domain to minimizing the energy of the output for an impulse of unit length.

When dealing with two dimensional spatio-temporal signals, it seems natural to define the 2D impulse as an impulse in the time domain (i.e. a signal concentrated at the origin of time), whose associated $l_2(\mathbf{Z})$ sequence in the spatial domain has unit norm. In the scalar case this gives

$$\delta(t,k) = \begin{cases} g(k) & t = 0 \\ 0 & t \neq 0, \end{cases}$$

with $g(k) \in l_2(\mathbf{Z})$ and such that $\sum_{k=-\infty}^{+\infty} |g(k)|^2 \leq 1$.

If we now apply this input to our system, which in input/output form is represented by

$$y(t,k) = \int_{-\infty}^{t} [\sum_{j=-\infty}^{\infty} w(t-\tau, k-j) u(\tau, j)] d\tau,$$

we get

$$y(t,k) = \sum_{j=-\infty}^{+\infty} w(t, k-j) g(j),$$

or, equivalently, if we take the Fourier transform in the spatial coordinate

$$y(t,\theta) = w(t,\theta) g(\theta).$$

We define the $\mathcal{H}_2^{2D}$ norm of a transfer function $w(t,\theta)$ to be

$$||w||_{\mathcal{H}_2^{2D}} = \sup_{||g||_2 \leq 1} \frac{||y||_2}{||g||_2}.$$

It can be proved [3] that

$$||w||_{\mathcal{H}_2^{2D}} = [\sup_{\theta \in [0,2\pi]} \int_0^{+\infty} w(t,\theta)^2 dt]^{\frac{1}{2}}.$$

An analogous result holds in the multivariable case too.

4 $\mathcal{H}_2^{2D}$ norm minimization

Let T_{zw} denote the transfer function from the exogenous inputs w to the regulated outputs z, in the standard configuration for robust control [2]. We want to solve the following optimization problem

$$\underline{\nu} = \inf_{K stabilizing} ||T_{zw}||_{\mathcal{H}_2^{2D}},$$

that is we want to find a controller, which is internally stabilizing and which minimizes the $\mathcal{H}_2^{2D}$-norm of the transfer matrix between w and z, T_{zw}.

Theorem 4.1 *[3] Let* T_{zw} *be as above. Then*

$$\inf_{K stabilizing} \sup_{\theta \in [0,2\pi]} ||T_{zw}||_2^2 = \sup_{\theta \in [0,2\pi]} \inf_{K stabilizing} ||T_{zw}||_2^2.$$

This theorem is key to solving our optimization problem. The auxiliary problem introduced in the theorem implies that solving our problem is equivalent to solving a standard $\mathcal{H}_2$ minimization problem for the 1D parametric system and then taking the supremum over θ for the overall cost. In [2] it is shown that this problem can be solved using the same results of classical finite dimensional $\mathcal{H}_2$ theory. The difference is that in this latter problem, the controller is no more unique.

Using the parametrization of stabilizing controllers introduced in Section 2, we can state our problem in the SISO case as a model matching problem

$$\nu(\theta) = \inf_{q \in l_1(\mathbf{Z})(s)} ||h(s,\theta) - u(s,\theta) q(s,\theta)||, \tag{4}$$

where h and u are fixed elements in $l_1(\mathbf{Z})(s)$. If u has no zeros on the imaginary axis for every θ, then the solution can be found using standard tools of finite dimensional systems.

There is one last thing that we want to stress. The controller that we obtain in this way has the same dynamical symmetries as the plant. That is, it is not only a distributed system, but it is also time/space invariant. The results in [2] show that, when the underlying dynamics of the system and the performance objective are spatially invariant, there is no performance loss in restricting the design to controllers which are themselves spatially invariant. This result is significant from the implementation point of view, since it implies that one only need to design the controller for a single actuator and all other controllers will be obtained by symmetry.

5 An example: $\mathcal{H}_2$ optimal control of an array of microcantilevers

In this section we will apply the theory we developped in the previous sections to an array of microcantilevers that are used in Atomic Force Microscopy and nano-scale manufacturing. A multicantilever structure consists of an array of microcantilevers that are connected to each other through a common base, and are individually actuated. The sensors are also integrated on each microcantilever. Hence, the physical and, consequently, the mathematical structure of the multicantilever model allows us to embed it in the class of spatially invariant systems. The following parameterized model is from [4]

$$\begin{aligned} \dot{x}(t,\theta) &= \begin{bmatrix} 0 & 1 \\ a(\theta) & 0 \end{bmatrix} x(t,\theta) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t,\theta) \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t,\theta), \end{aligned}$$

where $a(\theta) = a_{-1}e^{j\theta} + a_0 + a_1 e^{-j\theta}$ models the interaction of a cantilever with the sample and with the two closest cantilevers. The problem we want to solve is the minimization of the $\mathcal{H}_2^{2D}$-norm of the disturbance/output transfer function, T_{yd}. The problem can be stated as in (4), with $h = -\frac{2k_1k_2s + k_1^2 + k_2^2 a(\theta)}{s^2 + k_2 s - a(\theta) + k_1}$ and $u = -\frac{s^2 - a(\theta)}{s^2 + k_2 s - a(\theta) + k_1}$. k_i are the elements of the static feedback and observer, which are used to obtain a coprime factorization [3]. In the hypothesis $\frac{a_0}{2a_1} > 1$, u has no imaginary zeros for any $\theta \in [0,\ 2\pi]$. Therefore the $\mathcal{H}_2$ optimal problem has a solution. It turns out that the controller is described by the following transfer function

$$c(s,\theta) = \frac{n_5 s^5 + n_4 s^4 + n_3 s^3 + n_2 s^2 + n_1 s + n_0}{s^4 + d_3 s^3 + d_2 s^2 + d_1},$$

and the closed loop transfer function is

$$T_{yd} = \frac{n_5 s^5 + n_4 s^4 + n_3 s^3 + n_2 s^2 + n_1 s + n_0}{(s + \sqrt{a(\theta)})^2 (s^2 + k_2 s - a(\theta) + k_1)^2}.$$

The coefficients n_i and d_i are either rational or irrational functions of $a(\theta)$ (see [3]).

The improperness of C is due to the plant being strictly proper. By lowpass-filtering q, we can get a sequence of proper controllers, that achieve the infimum of the norm.

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