Notes on Magnetic Circuits
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Magnetic circuits play an important role in mechatronics as they appear in a number of actuators and sensors. Example applications are electric motors, magnetic bearings, electromagnetic latches, proximity sensors, and resolvers.

A good starting place is the analysis of the toroidal inductor shown in Figure 1. A ring or toroid of soft ferromagnetic materialof permeablility μ, is wound with *N* turns of wire. The toroid material may be any "soft" magnetic material that is magnetized when current is in the coil and then looses its magnetization when the current is turned off. Examples of soft ferromagnetic materials are iron, ferrite, 3% silicon-iron, 50% nickel-iron, 49% cobalt-iron. These materials differ in their maximum or "saturation" field levels and the degree to which they retain a residual magnetization.

A current, *I*, is applied to the wire and there is a corresponding voltage, *V*, on the terminals of the inductor. The current induces a field *H* (that has various names) and a magnetic field *B* approximately related to *H* by $B = \mu H$ provided that $|B| \leq B_{sat}$. The saturation magnetic field, B_{sat}, typically ranges from 0.4 Tesla (ferrite) to 2.4 Tesla (49%Co-Fe). Values of μ range from about 100 for Ferrite to 10,000 for 50%Ni-Fe. A typical range for motors and actuators is 1000-2000 for the commonly used silicon irons (silicon is added to increase the resistivity and reduce eddy-current losses).

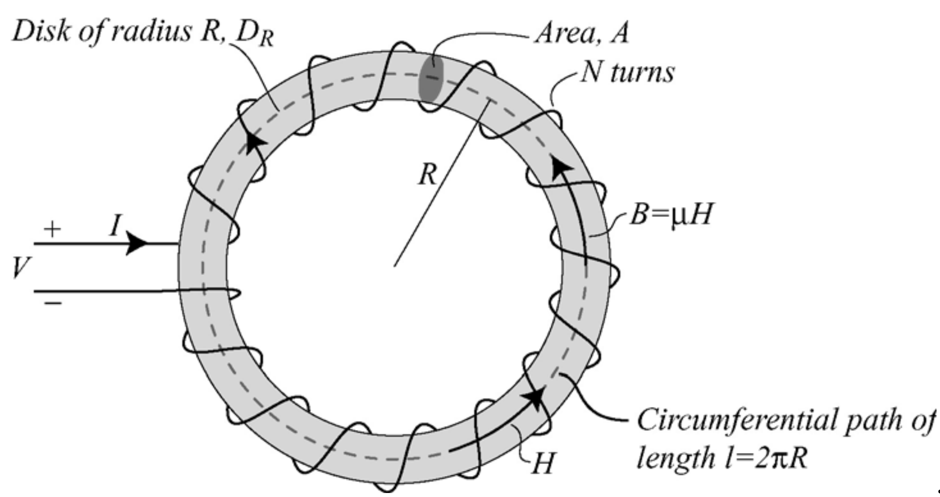

Figure 1. Toroidal Inductor

In the figure, we define the disk D_R having a radius of *R* to the midline of the toroid. The current in the windings passes through the disk out of the page, and returns into the page but not through the disk. Thus, there is a net current of *NI* through the disk and out of the page. By the symmetry of the problem we know that the scalar amplitude *H* of the vector field $\vec{H}$ is constant around the toroid as is the scalar amplitude *B* of the magnetic field $\vec{B}$. The cross section of the toroid has an area, *A*, as shown and the flux around the circuit is defined by

$$\phi \equiv BA \tag{1}$$

We take the positive sense for the flux to be counter-clockwise, the direction of flux when there is a positive current.

To understand this magnetic circuit, we derive the relationships between the various quantities we have defined.

From Maxwell's equations we have in differential form:

$$\nabla \times \vec{H} = \vec{J} \tag{2}$$

where $\vec{J}$ is the current density. In integral form we have

$$\oint_{\partial D_R} \vec{H} \cdot d\vec{l} = \iint_{D_R} \vec{J} \cdot d\vec{S} \tag{3}$$

Where ∂D_R denotes the boundary of the disk D_R. In words the line integral of the H field around the boundary of the disk is equal to the total current through the disk, NI. The positive sense of current is out of the page and the positive sense for the magnetic field is given by the right-hand rule and is counter-clockwise as we stated above.

Since H is uniform around the boundary of the disk, and since the length of the boundary is $l \equiv 2\pi R$ we have

$$H \underbrace{2\pi R}_{l} = NI \tag{4}$$

or

$$H = \frac{NI}{l} \text{ amps/meter} \tag{5}$$

Solving for the magnetic field and flux, we have

$$B = \mu H = \frac{\mu NI}{l} \tag{6}$$

and

$$\phi = BA = \frac{\mu NIA}{L} = \frac{\overbrace{Ni}^{\text{magneto-motive force (mmf)}}}{\underbrace{\left(\frac{l}{\mu A}\right)}_{\text{reluctance, } \mathcal{R}}} \tag{7}$$

We have defined the magneto-motive force (mmf) and the reluctance $\mathcal{R}$ in the equation. In magnetic circuits, magneto-motive force is analogous to voltage in electric circuits, reluctance is analogous to resistance, and flux is analogous to current. That is

$$I = \frac{V}{R} \text{ (Ohm's Law) is analagous to } \phi = \frac{mmf}{\mathcal{R}} \tag{8}$$

From Equation (7) above, we see that the reluctance is proportional to the length of iron, inversely proportional to the cross sectional area of the iron, and inversely proportional to the permeability of the iron. A similar relationship holds for the resistance of a conductor as shown in the figure below.

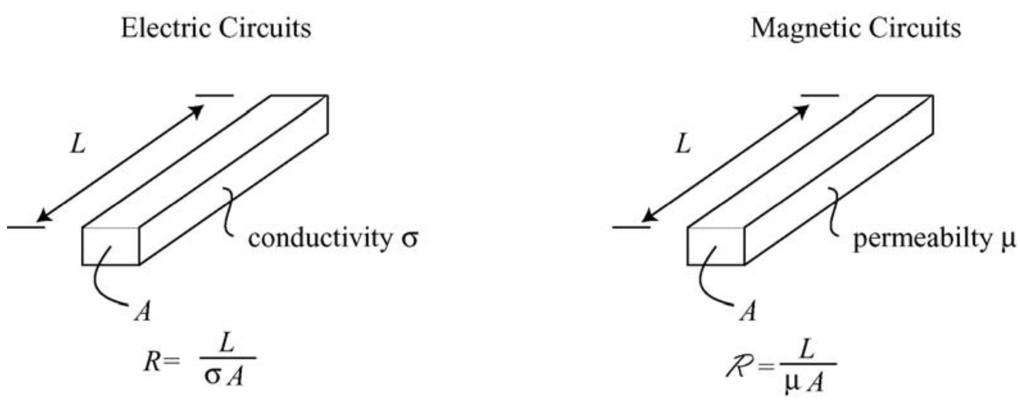

Figure 2. Analogy between resistance and reluctance.

Computing Voltage and Inductance

The voltage around a single turn in the winding of Figure 1 is given by another one of Maxwell's equations. Integrating the electric field around a single turn gives us

$$\begin{aligned} V_{\text{single turn}} &= \oint_{\text{Turn}} \vec{E} \cdot d\vec{l} = \iint_{\text{Area encircled}} -\frac{d\vec{B}}{dt} \cdot d\vec{A} \\ &= -\frac{d\phi}{dt} \text{ (this sign convention from physics is opposite ours)} \end{aligned} \tag{9}$$

For N turns we have, using our sign convention,

$$V = NV_{\text{single turn}} = N\frac{d\phi}{dt} \qquad (10)$$

Using the expression for flux from equation 1 yields

$$V = N\frac{d}{dt}\left(\frac{\mu NIA}{l}\right) = \underbrace{\left(\frac{N^2}{l/\mu A}\right)}_{\text{inductance, L}}\frac{dI}{dt} = \frac{N^2}{\mathcal{R}}\frac{dI}{dt} \qquad (11)$$

Example: Reluctances in series are summed

The figure below depicts a magnetic circuit having both iron and an air gap in the loop. The cross sectional area, *A*, is uniform around the circuit, the length of the iron and air gap are as indicated, and a positive current produces flux ϕ in the direction shown.

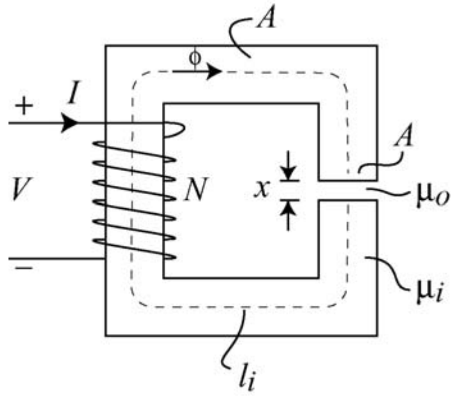

Figure 3. Inductor with air gap.

In this series connection, the reluctances of the iron and air add so that the circuit reluctance is

$$\mathcal{R} = \frac{l_i}{\mu_i A} + \frac{l_g}{\mu_o A} \qquad (12)$$

The flux depends on the current and the number of turns, *N*:

$$\phi = \frac{mmf}{\mathcal{R}} = \frac{NI}{\mathcal{R}} \qquad (13)$$

And the inductance at the coil terminals, using Equation 11, is

$$L = \frac{N^2}{\mathcal{R}} = \frac{N^2}{\frac{l_i}{\mu_i A} + \frac{l_g}{\mu_o A}} \tag{14}$$

Energy and Force Calculations

Since the power transferred into an inductor is $P = VI$, the total stored energy at time T in an inductor with zero initial current is

$$\begin{aligned} E &= \int_0^T VI dt \\ &= \int_0^T L \frac{dI}{dt} I dt \end{aligned} \tag{15}$$

Making the change of variable $u = I(t)$ such that $du = \frac{dI}{dt} dt$, we have

$$E = \int_0^{I(T)} Lu du = L \frac{u^2}{2} \bigg|_0^{I(T)} = \frac{1}{2} LI^2 \tag{16}$$

Referring again to the toroidal inductor reproduced below with inductance $L = \frac{\mu AN^2}{l}$.

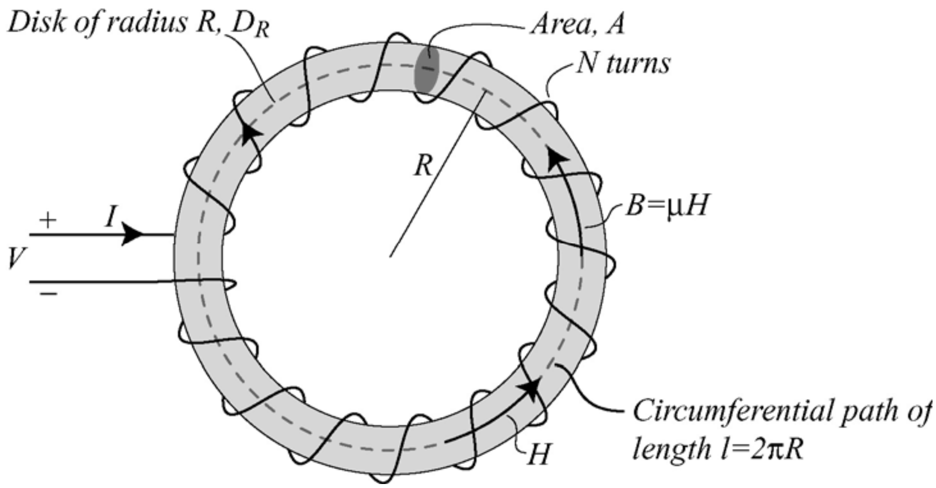

Figure 4. Toroidal inductor of Figure 1.

The energy stored is

$$E = \frac{1}{2}LI^2 = \frac{1}{2}\left(\frac{\mu AN^2}{l}\right)I^2 \tag{17}$$

Using Equation 6 for the magnetic field *B*, we can express the energy as an energy density times the volume of the magnetic material (we assume *R* is large).

$$E = \frac{1}{2}\left(\frac{\mu AN^2}{l}\right)I^2 = \underbrace{\left(\frac{\mu NI}{l}\right)^2}_{B^2}\frac{lA}{2\mu} = \underbrace{\frac{B^2}{2\mu}}_{\substack{\textit{energy}\\\textit{density}\\\text{(units of}\\\text{pressure)}}}\underbrace{lA}_{\substack{\textit{volume}\\\textit{of}\\\textit{toroid}}} \tag{18}$$

Note that the energy density only depends on the magnetic field and the material permeability, and has the units of pressure (recall that pressure times volume is work). It turns out that this is the same pressure (with μ replaced by μ_o in equations XXX) that applies a closing force to the air gap in Figure XXX. A more formal derivation of magnetic pressure is the following.

Force in a Gap

For the system of Figure 5 having a pivot that allow the gap to close, we can write down an energy balance equation. There the mechanical power delivered by pushing open the gap, plus the electrical power *VI*, is equal to the rate of change of energy stored in the inductor. To simplify the calculation, we short the coil terminal (apply zero voltage) so that the electrical power flow is zero. Since voltage is proportional to the rate of change

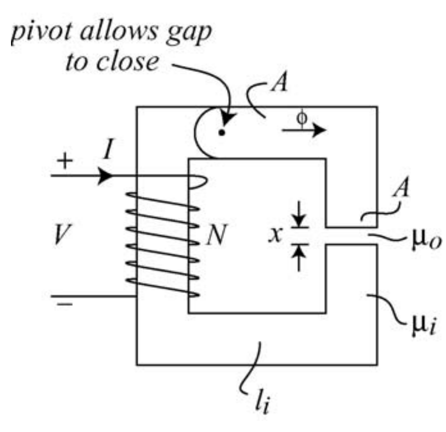

Figure 5. Magnetic circuit with pivot.

of flux, we have that the flux is constant and $\frac{d}{dx}\phi = 0$ when the coil is shorted. From Equations 13 and 14, we have

$$N\phi = LI \tag{19}$$

Both the inductance and current will depend on the gap *x*. Differentiating both sides with the shorted terminal assumption, we have

$$0 = \frac{d}{dx}L(x)I + L\frac{d}{dx}I(x) \tag{20}$$

Rearranging yields

$$L\frac{dI(x)}{dx} = -\frac{dL(x)}{dx}I \tag{21}$$

Since there is no power flowing into the electrical terminals, we have

$$\frac{dE}{dt} = \underbrace{\frac{d}{dx}\left(\frac{1}{2}L(x)I^2(x)\right)\frac{dx}{dt}}_{\substack{\text{rate of change of energy}\\ \text{in the inductor}}} = \underbrace{F\frac{dx}{dt}}_{\substack{\text{mechanical}\\ \text{power}}}$$

(22)

Setting $\frac{dx}{dt} = 1$ and differentiating the bracketed term with respect to *x* leads to

$$\left(\frac{1}{2}\frac{dL}{dx}I^2\right) + \left(LI\frac{dI}{dx}\right) = F \tag{23}$$

Where we have omitted the explicit dependence on x. Eliminating the *dI/dt* term using equation (21) leads to a partial cancellation in the two terms of Equation (23) and

$$F = -\frac{1}{2}\frac{dL}{dx}I^2 \tag{24}$$

We refer to Equation (25) as the inductance method for computing the force in a gap. The inductance can also be expressed as a function of the pivot angle in Figure 5. In that case, we would derivative in Equation (25) would be with respect to the pivot angle and the force would become the torque at the pivot. Such a calculation can be used to compute the torque in variable reluctance motors.

Pressure Method

Set $\mu_i = 0$ in Equation 14 (this is not necessary, but it simplifies the calculation). Differentiating Equation (14) with respect to x yields

$$\frac{dL}{dx} = -\frac{N^2 \mu_o A}{x^2} \qquad (25)$$

Applying the "inductance method" of Equation (24) gives

$$F = \frac{1}{2}\frac{N^2 \mu_o A}{x^2} I^2 = \frac{1}{2}\underbrace{\left(\frac{\mu_o NI}{x}\right)^2}_{B^2} \frac{A}{\mu_o} = \frac{B^2}{\underbrace{2\mu_o}_{\substack{\text{magnetic} \\ \text{pressure}}}} A \qquad (26)$$

We calculated the magnetic field *B* in the gap using a calculation similar to that leading to Equation (6). Looking at the last term in this equation, we express the force needed to keep the gap open as a pressure times an area. Where the pressure is proportional to the magnetic field squared.

Example: Compute the magnetic pressure in a gap when the magnetic field is equal to 1.6 Tesla (the approximate saturation flux density of 3% Si-Fe).

$$\frac{B^2}{2\mu_o} = \frac{(1.6T)^2}{2(4\pi \times 10^{-7} N / A^2)} = 1.02 MPa \qquad (27)$$

Converting to psi, we have

$$(1.02 \times 10^6) N / m^2 \times \frac{(0.0254 m / in)^2}{4.45 N / lb} = 148 \, psi \qquad (28)$$

Problems:

1. The saturation flux density of 50%Co-Fe is 2.4 Tesla. Compute the magnetic pressure in psi at this field level. Why is this material commonly used in aircraft motors and generators?

2. Go to www.fair-rite.com and look at the data sheet for the toroidal core #5943000310. They use the notation μ_i as the permeability relative to air. So that $\mu = \mu_i \mu_o$.
a) What are the permeability, magnetic path length, and cross sectional area for this part?
b) They define an inductance factor – what is that?
c) Is the inductance factor consistent with the numbers of part (a)?
d) How many turns are required to make an inductor of 4.4 millihenry (mH)?

e) At what current level in your inductor, will the core saturate? (Note: in the data sheet "Flux Density" is what we call B_{sat}).
f) How does the electrical conductivity of this material compare to that of pure iron?

3. Consider the magnetic circuit of Figure 5. Let
$\mu_i = 1000\mu_o$
$N = 100$
$l_i = 10cm$
$A = 1cm^2$
$x = 1mm$
a) If the saturation flux density in the iron is 2 T, what current is required to cause saturation?
b) What is the force in the gap at saturation?