Phasor Analysis of Linear Mechanical Systems
and Linear Differential Equations
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In this set of notes, we aim to imitate for linear mechanical systems and linear differential equations, the phasor analysis we learned for electric circuits.

Recall how we derived the complex impedance for an inductor. Starting with the differential equation for the V-I characteristic for an inductor

$$V = L\frac{d}{dt}I \tag{1}$$

we substitute complex sinusoids

$$V \to \hat{V}e^{j\omega t} \tag{2}$$

$$I \to \hat{I}\,e^{j\omega t} \tag{3}$$

So that equation (1) becomes

$$\hat{V}e^{j\omega t} = L\frac{d}{dt}\left(\hat{I}e^{j\omega t}\right) \tag{4}$$

Differentiating and solving yields

$$\hat{V} = j\omega L\hat{I} \tag{5}$$

and the impedance of the inductor is defined by

$$Z \equiv \frac{\hat{V}}{\hat{I}} = j\omega L \tag{6}$$

where "$\equiv$" denotes "defined equal to". Having done this calculation once, we see that we can jump directly from (1) to (5) by making the substitution

$$\frac{d}{dt} \to j\omega \tag{7}$$

Phasor Analysis of Linear Mechanical Systems

Consider a mechanical damper (a.k.a. shock absorber) which produces a velocity-dependent force according to the linear differential equation

$$f = b\frac{d}{dt}x \tag{8}$$

Making the substitution $\frac{d}{dt} \to j\omega$, we get

$$\hat{f} = j\omega b\hat{x} \tag{9}$$

And defining the mechanical impedance to be the ratio of force to displacement, we have

$$Z \equiv \frac{\hat{f}}{\hat{x}} = j\omega b \text{ (Newtons/meter)} \tag{10}$$

Note that the damper has a low stiffness at low frequencies, and a high stiffness at high frequencies. The units of impedance are Newtons/meter in mechanical systems and volts/amp = Ohms in electrical systems.

For a mass, m, we have

$$f = m\frac{d^2}{dt^2}x = m\left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right)x \tag{11}$$

Substituting $\frac{d}{dt} \rightarrow j\omega$ yields

$$\frac{\hat{f}}{\hat{x}} = (j\omega)^2 = -\omega^2 m \tag{12}$$

The impedance of a mass increases very rapidly with frequency. This explains why anvils and machine tools are massive. In summary, we have the following impedance properties for these basic mechanical components

Component	Impedance $\hat{f}/\hat{x}$ (Newtons/meter)
Mass, m	$-\omega^2 m$
Damper, b	$j\omega b$
Spring, k	k

Phasor analysis of interconnected linear mechanical systems

In analogy to the phasor analysis of electric circuits, we analyze mechanical systems with the following procedure

Step 1. Represent sinusoidal forces and displacements by phasors (complex numbers with the same amplitude and phase) denoted with hats as shown in Figure 2..

Step 2. Make the substitution $\frac{d}{dt} \rightarrow j\omega$ in component equations. In the case of mass, springs, dampers, forces, and displacements, this means we can substitute the complex impedances from table above in Figure 2. (Velocities, v, and accelerations, a, are related to displacements by differentiation, so $\hat{v} = j\omega\hat{x}$ and $\hat{a} = -\omega^2\hat{x}$. Once expressed as displacements, mechanical impedances can still be used.)

Step 3. Analyze using the laws of mechanics (treat the mechanical impedances as if they were springs with complex stiffnesses).

Step 4. Convert phasors to sinusoidal functions of time. (Since the phasor quantities have the amplitude and phase of the corresponding sinusoid we sometimes skip this step).

Example: For the mass-spring-damper system below with $f(t) = \cos(\omega t)$, solve for *x(t)*.

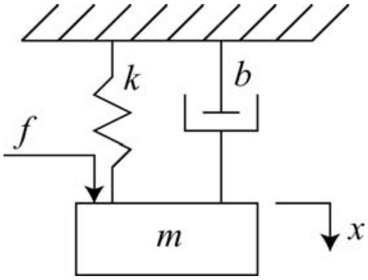

Figure 1. Mass-spring-damper system.

Following our procedure…

Step 1. Convert to phasor representation by putting hats on the time-dependent functions making them phasors (i.e. complex numbers).

Step 2. Make the substitution $\frac{d}{dt} \rightarrow j\omega$ in the differential equations. For mechanical systems this means we replace the mechanical parameters by the corresponding complex impedances. We verify that there are **only phasor forces and displacements** as impedances relate forces to displacements only.

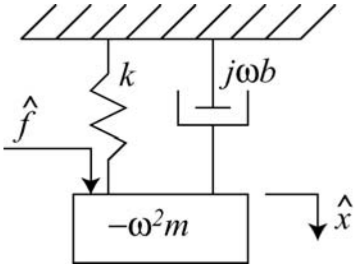

Figure 2. Phasor analysis of mass-spring-damper system.

Step 3. Apply the laws of mechanics to the complex quantities as though they are real stiffnesses (impedances). Since the components in the figure are in parallel, the forces produced by the spring, mass, and damper add to equal the applied force:

$$\hat{f} = k\hat{x} + j\omega b\hat{x} - \omega^2 m\hat{x} = \underbrace{\left(k + j\omega b - \omega^2 m\right)}_{impedance\ Z}\hat{x} \tag{13}$$

Since the applied force, $f\left(t\right) = \cos(\omega t)$, we have that

$$\hat{f} = 1 \tag{14}$$

Solving for $\hat{x}$ yields

$$\hat{x} = \frac{1}{k + j\omega b - \omega^2 m} \tag{15}$$

Step 4. Convert phasors to sinusoidal functions of time:

$$x(t) = \mathrm{Re}[\hat{x}e^{j\omega t}] \tag{16}$$

$$= \left|\hat{x}\right| \cos(\omega t + \angle\hat{x}) \tag{17}$$

$$= \frac{1}{\sqrt{(k - \omega^2 m)^2 + \left(\omega b\right)^2}} \cos(\left(\omega t - \mathrm{arctan}_2(\omega b / (k - \omega^2 m)\right) \tag{18}$$

Phasor Analysis of Linear Differential Equations

Any sinusoidally-forced linear differential equation with constant coefficients can be analyzed with phasors to find a sinusoidal steady state solution. Linear electric circuits and linear mechanical systems are special cases of systems described by such differential equations.

We describe this method by example:

Consider the sinusoidally-forced linear differential equation with constant coefficients:

$$\ddot{x} + 2\dot{x} + x = u(t); \quad u(t) = 3\sin(\omega t) \tag{19}$$

Steps 1 and 2. Applying step 1 and step 2 of the procedure we have

$$-\omega^2\hat{x} + j\omega 2\hat{x} + \hat{x} = \hat{u}; \quad \hat{u} = -3j \tag{20}$$

Step 3 becomes “apply the laws of algebra” rather than the laws of circuits or mechanics:

$$\frac{\hat{x}}{\hat{u}} = \frac{1}{-\omega^2 + j\omega 2 + 1}; \quad \hat{u} = -3j \tag{21}$$

The ratio on the left is called a "transfer function." It tells us how *u* affects *x*. In the case of circuits and mechanical systems, the transfer function is often an impedance (e.g. $Z = \frac{\hat{V}}{\hat{I}}$.) so that impedances are special cases of transfer functions. Solving for $\hat{x}$ yields.

$$\hat{x} = \frac{-3j}{-\omega^2 + j\omega 2 + 1} \tag{22}$$

Step 4. Converting hats to sinusoids… or not. The amplitude and phase of $\hat{x}$ is the same as the corresponding sinusoid so we can plot that directly. Or we can compute:

$$x(t) = \mathrm{Re}[\hat{x}e^{j\omega t}] \tag{23}$$

$$= |\hat{x}|\cos(\omega t + \angle\hat{x}) \tag{24}$$

$$= \frac{3}{\sqrt{(1-\omega^2)^2 + 4\omega^2}} \sin\left((\omega t - \pi/2 - \arctan(2\omega/(1-\omega^2))\right) \tag{25}$$

Note on stability: The underlying linear differential equation my be stable (having all characteristic roots with negative real part) or unstable. Correspondingly, the sinusoidal steady state computed using phasors will be stable or unstable. RLC circuits and mass-spring-damper systems are always stable since they conserve or dissipate energy.

Problems

Problem 1. Follow the derivation in equations (1) through (5) in these notes to derive the impedance of a mass, *m*.

Problem 2. Describe a mass-spring-damper system such that the phasor displacement of the mass is given by (22).

Problem 3. For equation (22), plot the amplitude and phase of the phasor $\hat{x}$.

Problem 4. For the system of Figure 3

(a) solve for the impedance $\frac{\hat{x}_1}{\hat{f}}$. (Hint: use a differential equation model rather than impedances. It is possible to use an impedance approach however)

What is the limiting impedance for $\omega \to 0$ and $\omega \to \infty$? Explain why these limits make sense.

(b) solve for the transfer function $\frac{\hat{x}_2}{\hat{f}}$. (Hint: use differential equations).

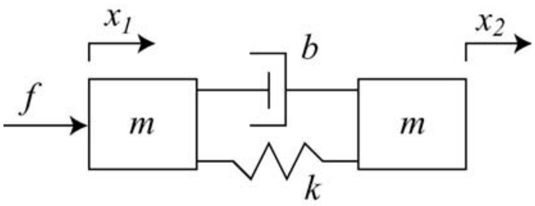

Figure 3.