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A Matched Nonlinearity for Phase Estimation of a PSK-Modulated Carrier

BRAD E. PADEN

Abstract—**A nonlinear phase estimator for a phase-shift keyed (PSK)-modulated carrier has been developed by Viterbi and Viterbi. Their analysis is extended, and an optimal or "matched" nonlinearity is derived for the estimator.**

I. INTRODUCTION

Packet transmission of digital data is common in satellite communication systems employing time-division multiple access (TDMA) techniques. These systems are usually characterized by communication terminals which have highly accurate and controlled timing that makes symbol synchronization from packet to packet possible. Phase accuracy, however, is almost impossible to maintain from one burst of a given user to the next. This follows from the fact that a small frequency error over a long packet can produce a substantial phase error.

Most conventional TDMA systems rely on a transmitted preamble of unmodulated carrier, or other predetermined signal, in each slot to estimate phase using phase-lock or correlation techniques. These approaches are inefficient when sufficient information is contained in the fully modulated carrier to obtain a phase reference. This motivates the development of a phase estimator which can extract a reference from a fully modulated carrier.

In [1] an open-loop nonlinear phase estimator was devised to estimate the phase of a phase-shift-keyed (PSK)-modulated carrier. This correspondence extends the results obtained there, and an optimal nonlinearity is derived for the open-loop phase estimator.

II. FORMULATION

Fig. 1 illustrates the structure of the phase estimator.[1] The input signal $y(t)$ is the m-PSK modulated carrier with additive white Gaussian noise (AWGN), which has the following form within a packet:

$$y(t) = \sqrt{2E_s/T}\sin\left(2\pi(f_0 + \Delta f)t + k_n(2\pi/m) + \vartheta\right) + n(t) \tag{1}$$

where

E_s	energy per symbol,
T	symbol duration,
k_n	random element of $\{1, 2, \cdots, m\}$ selected uniformly and held constant on intervals $[(n - (1/2)T, (n + (1/2)T)$,
f_0	carrier frequency,
Δf	carrier offset,
$n(t)$	AWGN with one-sided spectral density N_0,
ϑ	unknown carrier phase.

We wish to estimate ϑ given $y(t)$ on the interval $[-(N + (1/2))T, +(N + (1/2)T]$, where N is an integer.

After the modulated carrier is converted to baseband, the in-phase and quadrature channels are integrated and dumped providing the discrete-time sequence (x_n, y_n). Since accurate timing can provide symbol synchronization, we assume that the sampling following the integrator is synchronized with the incoming symbols. Also $y(t)$ is an m-PSK-modulated carrier with all symbols equally likely; thus the conditional density of (x_n, y_n) given that the carrier phase ϑ is invariant under rotations about the origin by angles that are integer multiples of $2\pi/m$. This symmetry is removed by the following nonlinear transformation, which is made in the dashed box in Fig. 1:

$$x'_n + iy'_n = F(\rho_n)e^{i(m\varphi_n)}, \tag{2}$$

where $\rho_n = \sqrt{x_n^2 + y_n^2}$, $\varphi_n = \tan^{-1}(x_n/y_n)$, and F is the nonlinearity to be chosen. The remaining blocks of the estimator yield the estimate of ϑ:

$$\hat{\vartheta} = \frac{1}{m}\tan^{-1}\left[\frac{\sum\limits_{n=-N}^{N} y'_n}{\sum\limits_{n=-N}^{N} x'_n}\right]. \tag{3}$$

This estimate has an m-fold ambiguity because it is derived from the modulated carrier. This problem can be overcome by a variety of methods, one of which is the use of differential encoding.

In the next section we solve the following problem. Given the form of the phase estimator in (3), find F, a map from the nonnegative real numbers to the real numbers (henceforth written $F\colon \mathbb{R}_+ \to \mathbb{R}$), such that $\operatorname{var}(\hat{\vartheta})$ is minimized.

III. CALCULATION OF THE MATCHED F

From [1] we have that the estimate $\hat{\vartheta}$ is unbiased and, for large N, $\operatorname{var}(\hat{\vartheta})$ satisfies the following proportionality (the constant of proportionality is independent of F)

$$\operatorname{var}(\hat{\vartheta}) \propto \frac{\left(E(F^2(\rho)) - [E(F(\rho)\cos\epsilon')]^2\right)[1 - S_N(2\Delta')] + \left(E(F^2(\rho)) - E[F^2(\rho)\cos 2\epsilon']\right)S_N(2\Delta')}{[E(F(\rho)\cos\epsilon')]^2 S_N(\Delta')}, \tag{4}$$

where

$$S_N(\Delta') = \frac{1}{2N+1}\left[\frac{\sin[(2N+1)\pi\Delta']}{\sin(\pi\Delta')}\right],$$

$$\epsilon' = m\epsilon,$$

and

$$\Delta' = m(\Delta f),$$

with

$$p(\rho,\epsilon) = \frac{\rho}{2\pi\sigma^2}\exp\left\{-\left\{\frac{\rho^2}{2\sigma^2} + \frac{\gamma}{2} - \frac{\rho\sqrt{\gamma}\cos\epsilon}{\sigma}\right\}\right\}, \tag{5}$$

$$\sigma^2 = (N_0/2),$$

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[1] The amplitudes of the mixing signals in this diagram differ from those in [1]. This is done so that the probability density of (x_n, y_n) is independent of T.

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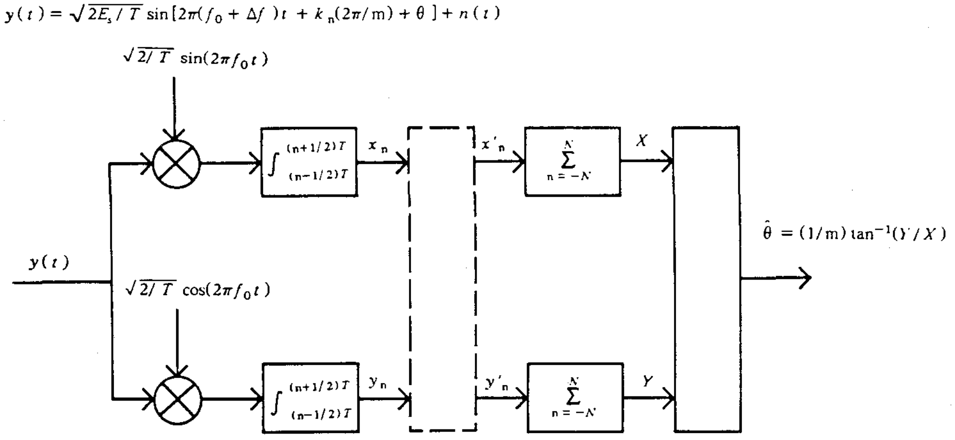

Fig. 1. Estimator structure.

and

$$\gamma = 2E_S/N_0.$$

Thus we choose F to minimize the right hand side of (4).

The calculation is as follows. From (4),

$$\operatorname{var}(\hat{\vartheta}) \propto \frac{1}{S_N(\Delta')}\left[S_N(2\Delta') - 1 + \frac{E\big(F^2(\rho)(1 - S_N(2\Delta')E(\cos 2\epsilon'|\rho))\big)}{\{E[F(\rho)E(\cos\epsilon'|\rho)]\}^2}\right]. \quad (6)$$

Define

$$h(\rho) = 1 - S_N(2\Delta')E(\cos 2\epsilon'|\rho)$$

and

$$q(\rho) = E(\cos\epsilon'|\rho).$$

Thus (4) may be rewritten as

$$\operatorname{var}(\hat{\vartheta}) \propto \frac{1}{S_N(\Delta')}\left[S_N(2\Delta') - 1 + \frac{E[F^2(\rho)h(\rho)]}{E[F(\rho)q(\rho)]^2}\right]. \quad (7)$$

Now from (5) and Bayes' rule we see that, conditioned on ρ, ϵ has the Tikhonov distribution with density

$$p(\epsilon|\rho) = \frac{\exp\left(\frac{\rho\sqrt{\gamma}}{\sigma}\cos\epsilon\right)}{\int_{-\pi}^{\pi}\exp\left(\frac{\rho\sqrt{\gamma}}{\sigma}\cos\epsilon\right)d\epsilon} = \frac{\exp\left(\frac{\rho\sqrt{\gamma}}{\sigma}\cos\epsilon\right)}{2\pi I_0\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)} \quad (8)$$

where I_m is the mth-order modified Bessel function of the first kind [2]. So we calculate

$$h(\rho) = 1 - \frac{S_N(2\Delta')I_{2m}\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)}{I_0\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)} \quad (9)$$

and

$$q(\rho) = \frac{I_m\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)}{I_0\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)}. \quad (10)$$

Since $S_N(2\Delta') \le 1$, and $I_0(r) > I_m(r) \ge 0$ $\forall r \ge 0$, $m \in \{1, 2, 3, \cdots\}$, we have $h(\rho) > 0$ $\forall\rho \ge 0$. Now define for a, b: $\mathbb{R}_+ \to \mathbb{R}$:

$$\langle a, b\rangle = E[a(\rho)b(\rho)h(\rho)] \quad (11)$$

where the integration in (11) is with respect to the marginal distribution for ρ derived from (5). Since $h(\rho) > 0$ $\forall\rho \ge 0$, $\langle\cdot,\cdot\rangle$ is an inner product on $L^2 \equiv \{a\colon \mathbb{R}_+ \to \mathbb{R} | \langle a, a\rangle < \infty\}$. Moreover, (7) can be written

$$\operatorname{var}(\hat{\vartheta}) \propto \frac{1}{S_N(\Delta')}\left[S_N(2\Delta') - 1 + \frac{\langle F, F\rangle}{\left\langle F, \frac{q}{h}\right\rangle^2}\right]. \quad (12)$$

So by Schwarz' inequality, the $F \in L^2$ which minimizes $\operatorname{var}(\hat{\vartheta})$ is

$$F(\rho) = \gamma\frac{q(\rho)}{h(\rho)} = \lambda\frac{I_m\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)}{I_0\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right) - S_N(2\Delta')I_{2m}\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)}, \quad (13)$$

where λ is an arbitrary nonzero constant.

Since Δf is almost always unknown, a case of particular interest is $\Delta f = 0$ for which $S_N(\Delta') = 1$. If, in addition, we take $m = 1$, that is, an unmodulated carrier, (13) becomes

$$F(\rho) = \lambda\frac{I_1\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)}{I_0\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right) - I_2\left(\frac{\rho\sqrt{\gamma}}{\sigma}\right)} = \lambda\frac{\rho\sqrt{\gamma}}{2\sigma}. \quad (14)$$

Since $F(\rho)$ is now linear, the scale factor $\lambda(\sqrt{\gamma}/2\sigma)$ on the right side of (14) cancels in (3), and the estimator reduces to the maximum-likelihood[2] estimate of an unmodulated carrier [3].

Fig. 2 depicts F versus ρ for $\Delta f = 0$, $m = 4$, at several values of E_s/N_0. The values of λ used in the figure are such that $\lim_{\rho\to\infty} F(\rho)/\rho$ is the same for each curve. The limiting behavior of F is discussed further in the next section.

IV. THE ASYMPTOTIC NONLINEARITY FOR LARGE AND SMALL E_s/N_0

For receivers operating at extreme values of E_s/N_0, the optimal nonlinearity may be approximated closely by a monomial ρ^k,

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[2] This is also the minimum mean-square error estimate by the symmetry of both the quadratic cost function and the conditional density $p(\theta|\{(x_n, y_n)\}_{n\in\{-N,\cdots,N\}})$.

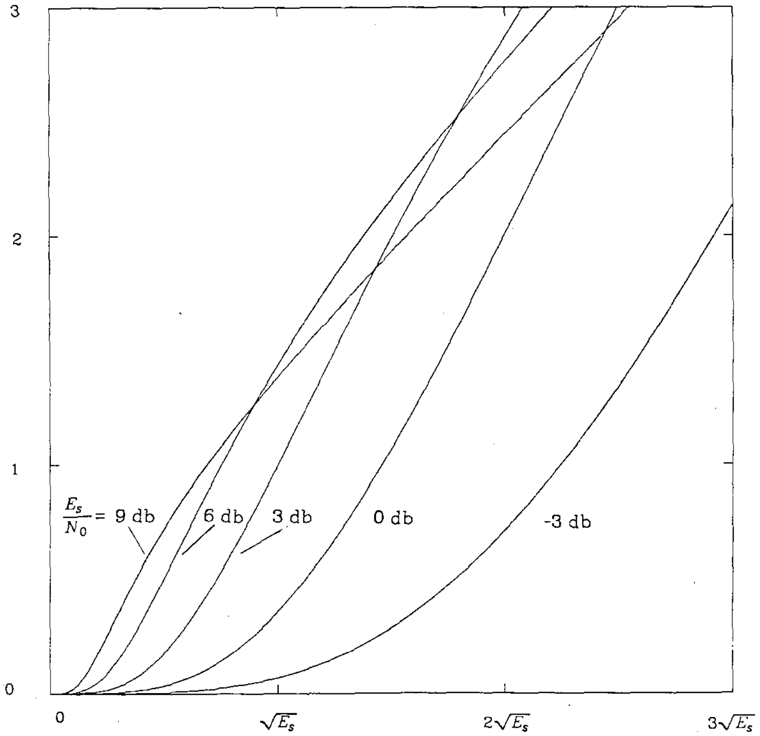

Fig. 2. F versus ρ for $m = 4$, $\Delta f = 0$.

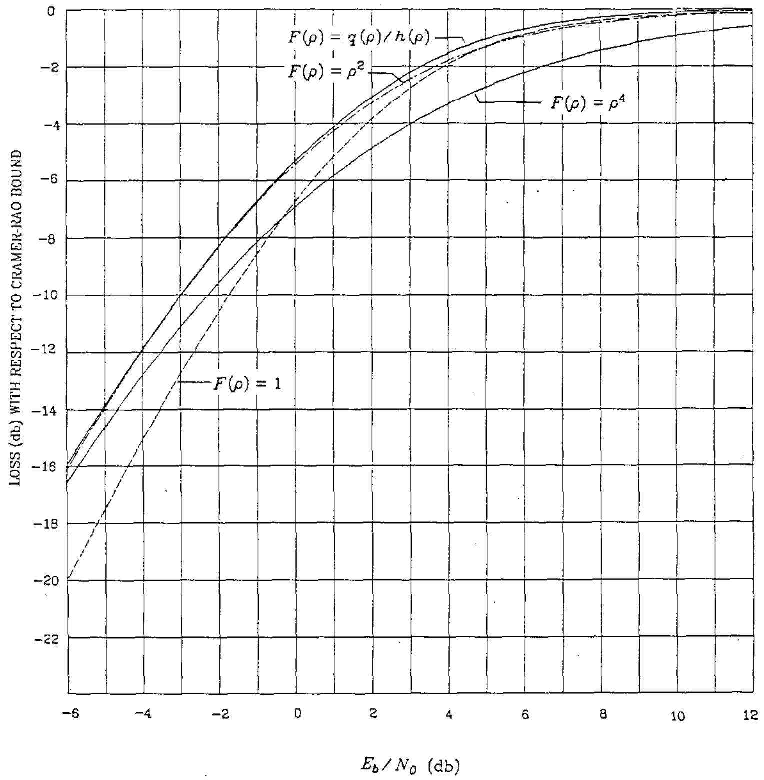

Fig. 3. Performance of phase estimator as function of E_b/N_0 for $m = 4$, $\Delta f = 0$.

where k is an integer. From [2] we have

$$I_m(r) = \frac{e^r}{\sqrt{2\pi r}}\left[1 - \left(\frac{4m^2 - 1}{8r}\right) + \frac{(4m^2 - 1)(4m^2 - 9)}{2!(8r)^2} + \cdots\right], \quad (15)$$

which we use for approximating $I_m(r)$ when $r \gg 1$. For small values of r we use

$$I_m(r) = \left(\frac{r}{2}\right)^m\left[\frac{1}{m!} + \frac{(r/2)}{(m+1)!} + \cdots\right]. \quad (16)$$

From (15) and the fact that $\rho = O(\sqrt{E_s})$, we have that for $E_s/N_0 \gg 1$ and $S_N(2\Delta') = 1$ the optimal nonlinearity (for appropriate choice of λ) is

$$F(\rho) = \rho, \tag{17}$$

and for $S_N(2\Delta') < 1$, $E_s/N_0 \gg 1$:

$$F(\rho) = 1. \tag{18}$$

Similarly, from (16), for $E_s/N_0 \ll 1$ and arbitrary $S_N(2\Delta')$,

$$F(\rho) = \rho^m. \tag{19}$$

The primary advantage of using these monomial approximations of F is that no measurement of E_s/N_0 is necessary.

For $E_s/N_0 \gg 1$, the optimal nonlinearity is dependent on the frequency offset (see (17) and (18)), whereas at low E_s/N_0 it is not. This can be understood heuristically by noting that, in the limit as $E_s/N_0 \to 0$, the phase errors due to the frequency offset are negligible with respect to those due to the noise. At high E_s/N_0, however, the opposite is true and a dependence on Δ' appears.

V. PERFORMANCE OF MONOMIAL NONLINEARITIES FOR MODERATE E_s/N_0

To compare the performance of monomial nonlinearities to that of the optimal, (12) is integrated numerically for $m = 4$(QPSK) (quadrature phase shift keying), $\Delta f = 0$, and $F(\rho) = q(\rho)/h(\rho)$, the optimal nonlinearity. The performance (relative to the Cramer–Rao bound [1]) of the optimal and several monomial nonlinearities as a function of energy per bit divided by the one-sided spectral noise density is shown in Fig. 3. The fact that the performance of the nonlinearity $F(\rho) = \rho^2$ is very close to the optimal is important. Although a small gain in performance occurs when the optimal nonlinearity is used near $E_b/N_0 = 4$, the optimal nonlinearity requires E_b/N_0 dependent scaling of the received signal and is therefore sensitive to automatic gain control levels. From (3) we can see for monomial $F(\rho)$ that any scaling of ρ simply cancels in the numerator and denominator. This is a very nice property and a strong motivation for using the ρ^2 nonlinearity. We can conclude that for the $m = 4$ case (QPSK), ρ^2 is an excellent choice of nonlinearity in agreement with the claims of [1].

VI. CONCLUSION

Using the intermediate result of [1] we have derived an optimal nonlinearity for the estimator in Fig. 1. This nonlinearity is dependent on E_s/N_0 so that the signal-to-noise ratio must be estimated if optimal phase estimation is required at all noise levels. However, if it is known that E_s/N_0 is extreme, high or low, then there are monomial approximations for the nonlinearity which (asymptotically) do not depend on the signal-to-noise ratio. For the $m = 4$, $\Delta f = 0$ case we have shown that ρ^2, the nonlinearity proposed in [1], nearly achieves the upper bound in performance given by the optimal nonlinearity, and for practical reasons it is ideal for implementation.

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On the Distribution of de Bruijn CR-Sequences

T. ETZION

Abstract—**It is shown that the number of de Bruijn sequences of order n and linear complexity c is not a multiple of four for every n and c.**

In [1] de Bruijn sequences and their complexities are presented, and in [2] and [3] de Bruijn CR-sequences are investigated. Chan *et al.* [1] proved that the linear complexity c of a de Bruijn sequence of order n is an integer between $2^{n-1} + n$ and $2^n - 1$. They conjectured that the number of de Bruijn sequences of order $n > 3$ and linear complexity c, $\gamma(c, n)$, is a multiple of four, i.e., $\gamma(c, n) \equiv 0$ (modulo 4).

Let $\delta(c, n)$ denote the number of de Bruijn CR-sequences of order n and of linear complexity c. It is well-known that $\gamma(c, n) \equiv 0$ (modulo 4) if and only if $\delta(c, n) \equiv 0$ (modulo 4). Since for even $n \geq 4$ there are no de Bruijn CR-sequences, we have $\gamma(c, n) \equiv 0$ (modulo 4) for even n. For odd n there exist de Bruijn CR-sequences, and the following results have been obtained. Etzion and Lempel [2] showed that for even c $\delta(c, n) = 0$, and therefore $\gamma(c, n) \equiv 0$ (modulo 4). For $n \geq 4$, $\gamma(2^n - 1, n) \equiv 0$ (modulo 8) and for $k \geq 3$, $\gamma(2^{2k} - 1, 2k) \equiv 0$ (modulo 16). Etzion [3] showed that, if $2^{n-1} < c < 2^{n-1} + 2^{n-2}$, then $\delta(c, n) \equiv 0$ (modulo 4), and therefore $\gamma(c, n) \equiv 0$ (modulo 4).

The two de Bruijn sequences of order 3 are CR-sequences.

The complexities distribution of de Bruijn CR-sequences of order 5 were easily obtained:

c	$\delta(c,5)$
23	4
25	8
27	12
29	8
31	32

For any other c not in the table, $\delta(c, 5) = 0$.

For $n = 7$, the characterization of de Bruijn CR-sequences [2], [3] and the Games and Chan [4] algorithm for computing the complexity of a sequence of length 2^n were used for a computer computation of $\delta(c, 7)$:

c	$\delta(c,7)$	c	$\delta(c,7)$	c	$\delta(c,7)$	c	$\delta(c,7)$
71	448	85	236	99	11802	113	1102220
73	8	87	284	101	20258	115	2116456
75	168	89	844	103	31144	117	4210074
77	24	91	1620	105	72250	119	8328830
79	88	93	2560	107	143238	121	16875998
81	40	95	6424	109	285742	123	33706580
83	224	97	7488	111	559216	125	67480984
						127	131815424

For any other c, $\delta(c, 7) = 0$.

It is now clear that for $c = 99$, 101, 105, 107, 109, 117, 119, and 121, $\delta(c, 7)$ is not a multiple of four and hence $\gamma(c, 7)$ is not a multiple of four.

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