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Globally asymptotically stable 'PD+' controller for robot manipulators

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We describe a globally stable tracking controller for robot manipulators. The controller is an extension of Takegaki and Arimoto's position controller to the tracking case where a theorem of Matrosov is used to prove its stability. An attractive feature of this controller is its resemblance to the computed torque controller with the inertia matrix outside the position and velocity feedback loops. Thus, our controller is decomposed into an inner PD loop and an outer dynamic compensation loop. This structure allows the simple PD computations to be run at a higher speed than the dynamic compensation loop in digital implementations.

1. Introduction

In this paper we present a simple globally stable tracking control scheme for robot manipulators. This scheme is a tracking controller which, from a block diagram point of view, is a variation of the popular computed torque scheme. The advantage of the scheme is that the manipulator inertia matrix is *outside* the position and velocity feedback loops. This allows the decomposition of the control scheme into decoupled PD joint control *plus* a higher level dynamics compensation loop. We refer to this controller as a 'PD+' controller. The higher level loop feeds forward nominal joint torques and compensates for gravity and forces which are Coriolis-like (see Fig. 1).

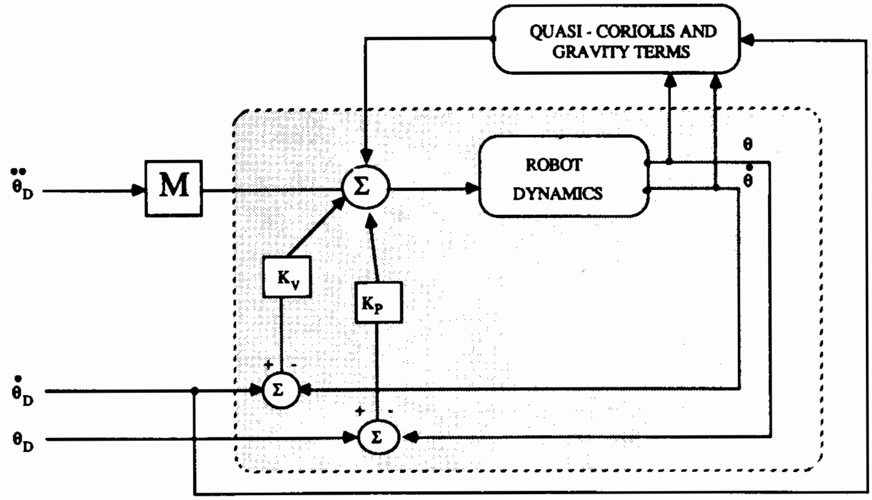

Figure 1. 'PD+' control scheme.

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The advantage of this structure is that the PD controllers can be run much faster in digital implementations. (It is generally believed that the position and velocity feedback loops must be updated at a higher rate than the dynamic compensation terms in digital robot controllers. This is motivated by the fact that the feedback terms have high gains.)

To provide the background for our work, we review some of the presently available control schemes. The computed torque controller is the globally stable controller which is easiest to analyse as we can use simple linear theory. Robustness of this scheme in the presence of bounded disturbance torques, measurement noise and parameterized variations in the manipulator dynamics has also been shown by Spong and Vidyasagar (1985). Other schemes with stability proofs are Lyapunov designs. There are several of these. Variable structure controllers have been developed by Slotine (1985), Slotine and Sastry (1983), and Paden and Sastry (1987). Robust controllers which are similar to the variable structure controllers, but not having discontinuous control laws, have been proposed by Corless and Leitmann (1984) and Slotine (1985). Robust controllers typically have somewhat complicated stability proofs, but the controllers themselves tend to be rather simple and have computational advantages when the computation of manipulator dynamics is expensive.

A position control scheme, which is beautifully simple, is due to Takegaki and Arimoto (1981). This scheme is designed by deriving a Lyapunov function from the mechanical energy of the manipulator. It appears that this scheme had not been extended to the tracking control problem due to technical problems in proving stability. In the Takegaki and Arimoto scheme, stability is proved using LaSalle's theorem for autonomous systems. However, when the desired position of the manipulator is time-varying, this theorem is no longer applicable, thus the technical difficulty. Recently, Slotine and Li (1987) cleverly overcame this difficulty by using some ideas from variable structure control. They introduced a complete adaptive control law which has a tracking control version as well. This scheme has the inertia matrix inside the 'D' part of the PD feedback loop (see Fig. 2). This is not necessary in

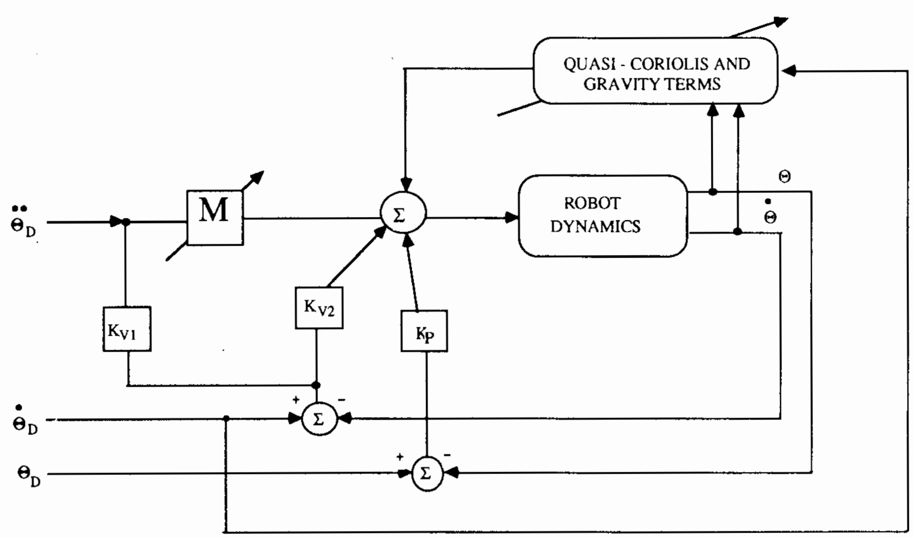

Figure 2. Slotine and Li's adaptive scheme.

the tracking case, as we show here, and the resulting controller can be designed to have independent PD joint control with a high level outer control loop for dynamic compensation. Recently, Bayard and Wen (1987) (see also Wen and Bayard 1987) have studied both the adaptive and tracking control of robot manipulators. Their work is exceptional in that they propose some control schemes which are *exponentially stable*. Their analysis technique differs from ours and that of Slotine and Li. We use a theorem of Matrosov to prove stability, whereas Slotine and Li integrate some ideas from variable structure control and Bayard and Wen use some technical lemmas and carefully chosen Lyapunov functions. These are all Lyapunov-based approaches, but differ in interesting ways.

In this paper we develop our controller as follows. In § 2 we introduce our notation and derive the manipulator dynamics, paying careful attention to the structure of the fictitious forces derived from the inertia tensor. In § 3 we re-derive Takegaki and Arimoto's controller and in § 4 we develop our controller. We demonstrate the stability of the control scheme in § 5 using a simulation of a two-link manipulator. Section 6 contains our conclusions, and the complete stability proof based on a theorem of Matrosov is contained in the Appendix.

2. Manipulator dynamics

In this section we derive the dynamics of an m-joint rigid-link robot manipulator using tensor notation. This general construction is important as it preserves the structure of the dynamics very well; the fictitious forces are described explicitly in terms of the inertia tensor and their multilinearity is clear. We use the following notation.

$\mathbb{R}$	real numbers
$\mathbb{R}_+$	non-negative real numbers
C^n	continuously differentiable n times
$f^{(n)}(t)$	nth time derivative of $f(t)$
$d(x, \Omega)$	distance from the point x to the set Ω
$[M]$	the matrix whose components are the same as the covariant tensor M of order 2 (in other words, $\alpha^T[M]\beta \triangleq M(\alpha, \beta)$)
D_x	derivative with respect to x
A_{ij}	the alternation mapping which skew-symmetrizes a tensor in its ith and jth arguments
$F(\cdot)$	the map $F(\cdot): \alpha \to F(\alpha)$
B_α	open ball of radius α centred at the origin
$\bar{S}$	closure of the set S

Let $\theta \in \mathbb{R}^m$ be the vector of manipulator joint displacements and denote the configuration-dependent inertia tensor of the robot manipulator by $M(\theta)$. We make the *Assumption* 1 that $M(\theta)$ is symmetric, positive definite, twice continuously differentiable, and its maximum and minimum singular values are bounded and bounded away from zero, respectively. Denote the components of $M(\theta)$ by $M(\theta)_{ij}$. That is,

$$M(\theta)(\alpha, \beta) = M(\theta)_{ij}\alpha^i\beta^j \tag{2.1}$$

where α^i are the components of $\alpha \in \mathbb{R}^n$, and the summation over i and j on the right-hand side is implied. The derivative of $M(\theta)$ with respect to θ is a covariant tensor of order 3 with components

$$D_\theta M(\theta)_{ijk} = \frac{\partial M(\theta)_{ij}}{\partial \theta^k} \tag{2.2}$$

We define

$$D_\theta M(\theta)(\alpha, \beta)(\gamma) \triangleq D_\theta M(\theta)_{ijk}\alpha^i \beta^j \gamma^k \tag{2.3}$$

and

$$D_\theta^2 M(\theta)(\alpha, \beta)(\gamma)(\delta) \triangleq \frac{\partial M(\theta)_{ij}}{\partial \theta^k \partial \theta^l} \alpha^i \beta^j \gamma^k \delta^l \tag{2.4}$$

Thus, $D_\theta M(\theta)$ is linear in each of the three arguments α, β and γ, as is $D_\theta^2 M(\theta)$ in its four. In addition, let $G(\theta)$ denote the potential energy due to gravity for the configuration θ. We make the *Assumption* 2 that $G(\theta)$ is continuously differentiable. The lagrangian for a manipulator is therefore

$$L = \tfrac{1}{2} M(\theta)(\dot\theta, \dot\theta) - G(\theta) \tag{2.5}$$

The joint forces are given by

$$F = \frac{d}{dt} D_{\dot\theta} L - D_\theta L \tag{2.6}$$

In this formalism F is a linear operator (covariant tensor of order 1) which acts on a displacement to do work. From (2.5) and (2.6) we have

$$F(\cdot) = \frac{d}{dt} M(\theta)(\dot\theta, \cdot) - \tfrac{1}{2} D_\theta M(\theta)(\dot\theta, \dot\theta)(\cdot) + D_\theta G(\theta)(\cdot) \tag{2.7}$$

$$= M(\theta)(\ddot\theta, \cdot) + D_\theta M(\theta)(\dot\theta, \cdot)(\dot\theta) - \tfrac{1}{2} D_\theta M(\theta)(\dot\theta, \dot\theta)(\cdot) + D_\theta G(\theta)(\cdot)$$

$$= M(\theta)(\ddot\theta, \cdot) + \tfrac{1}{2} D_\theta M(\theta)(\dot\theta, \cdot)(\dot\theta) - A_{23} D_\theta M(\theta)(\dot\theta, \dot\theta)(\cdot) + D_\theta G(\theta)(\cdot) \tag{2.8}$$

where

$$A_{23} D_\theta M(\theta)(\alpha, \beta)(\gamma) \triangleq \tfrac{1}{2}[D_\theta M(\theta)(\alpha, \beta)(\gamma) - D_\theta M(\theta)(\alpha, \gamma)(\beta)] \tag{2.9}$$

and $D_\theta G(\theta)(\alpha) \triangleq \dfrac{\partial G}{\partial \theta^i}\alpha^i$. Observe that $A_{23} D_\theta M(\theta)(\dot\theta, \dot\theta)(\cdot)$ is a force which does no work on the system since $A_{23} D_\theta M(\theta)(\dot\theta, \dot\theta)(\dot\theta) = 0$. This structure has been exploited by Koditschek (1984) and Slotine and Li (1987). Equation (2.8) is a particularly convenient expression of the dynamics and is the form that we will use.

3. Position control

We begin the development of our control scheme by reviewing the position control scheme of Takegaki and Arimoto (1981). Their scheme relies on a very simple and appealing idea: compensate, through position feedback, for the gravity forces and then add joint forces which are minus the gradient of some desired potential (having a minimum at the desired set-point). By adding damping, the convergence of the manipulator state to the desired set-point is guaranteed.

We re-derive their position controller in a slightly different way to motivate the analysis of our tracking control scheme and to accustom the reader to our notation. The position control problem is: given a desired configuration θ_d, find a control law such that the manipulator state, $[\theta^T \quad \dot{\theta}^T]^T$, converges to $[\theta_d^T \quad 0^T]^T$. Following Takegaki and Arimoto we choose as the Lyapunov function candidate

$$V(\theta, \dot{\theta}) = \tfrac{1}{2}M(\theta)(\dot{\theta}, \dot{\theta}) + \tfrac{1}{2}K_p(\theta - \theta_d, \theta - \theta_d) \tag{3.1}$$

where K_p is symmetric and positive definite (*Assumption* 3). Next, the joint forces are chosen so that $\dot{V}$ is negative. Differentiating (3.1) yields

$$\dot{V} = M(\theta)(\ddot{\theta}, \dot{\theta}) + \tfrac{1}{2}D_\theta M(\theta)(\dot{\theta}, \dot{\theta})(\dot{\theta}) + K_p(\theta - \theta_d, \dot{\theta}) \tag{3.2}$$

Using the manipulator dynamics (2.8) to eliminate the $\ddot{\theta}$ term we obtain

$$\dot{V} = F(\dot{\theta}) + K_p(\theta - \theta_d, \dot{\theta}) - D_\theta G(\theta)(\dot{\theta}) \tag{3.3}$$

Note that all the fictitious forces cancelled in this calculation. This is peculiar to the position control problem and does not happen in the tracking control case as we will see in the next section. Choose joint forces of the form

$$F = -K_v(\dot{\theta}, \cdot) - K_p(\theta - \theta_d, \cdot) + D_\theta G(\theta)(\cdot) \tag{3.4}$$

where K_v is symmetric and positive definite (*Assumption* 4). (K_p can be replaced by a general potential function which is positive definite in the position error as done by Takegaki and Arimoto 1981.) Thus, we have

$$\dot{V} = -K_v(\dot{\theta}, \dot{\theta}) \tag{3.5}$$

which is only positive semi-definite in the state error. Thus, we cannot conclude stability immediately. LaSalle's theorem can be used (e.g. see Vidyasagar 1978) to show convergence if we verify that the largest invariant set in $\{[\theta^T \quad \dot{\theta}^T]^T \mid \dot{V} = 0\} = \{[\theta^T, \dot{\theta}^T]^T \mid \dot{\theta} = 0\}$ is the desired state $[\theta_d \quad 0]^T$. From (2.8) and (3.4) we have

$$\ddot{\theta} = -[M(\theta)]^{-1}[K_p](\theta - \theta_d) \quad \forall t \text{ such that } \dot{V}(t) = 0 \tag{3.6}$$

Recall that $M(\theta)$ and K_p are symmetric positive definite and so, if $\theta \neq \theta_d$ and $\dot{V} = 0$, then $\ddot{\theta} \neq 0$ (i.e. the state moves off the set $\{[\theta^T \quad \dot{\theta}^T]^T \mid \dot{V} = 0\}$). Thus, $[\theta_d^T \quad 0^T]^T$ is the only invariant point in the set $\{[\theta^T \quad \dot{\theta}^T]^T \mid \dot{V} = 0\}$. It follows, by LaSalle's theorem, that the manipulator state converges to $[\theta_d^T \quad 0^T]^T$. LaSalle's theorem is a key element in the stability proof given by Takegaki and Arimoto. We cannot, however, use this theorem

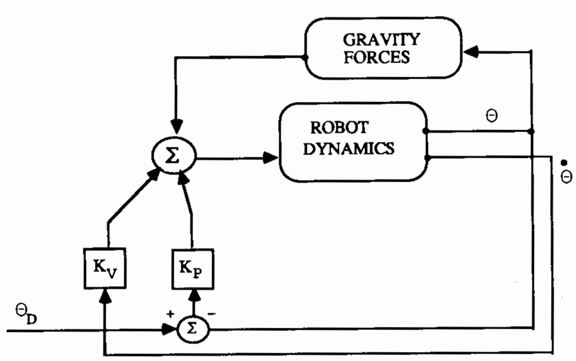

Figure 3. Takegaki and Arimoto's scheme.

in the tracking control case as the error dynamics are no longer described by an autonomous differential equation.

This control scheme is blocked out in Fig. 3. The feedback is rather simple except for the gravity compensation terms.

4. Tracking control

The control of high-speed manipulators requires more than the position control law of the last section. Rather than specifying a desired point in the manipulator state-space we now specify an entire trajectory and design a controller which will guarantee that the manipulator state converges to this trajectory. In this section we give a direct extension of Takegaki and Arimoto's scheme. The resulting controller can also be viewed as a reconfiguration of the computed torque control law with the inertia matrix *outside* the position and velocity feedback loop. Our stability proof relies on a theorem of Matrosov (see Rouche *et al.* 1977).

The tracking control problem is the following. Given a bounded desired position trajectory $\theta_d(t)$ which is twice continuously differentiable and has bounded first and second derivatives (*Assumption* 5), find joint forces which will guarantee that the manipulator state, $[\theta(t)^T \quad \dot{\theta}(t)^T]^T$, converges to $[\theta_d(t)^T \quad \dot{\theta}_d(t)^T]^T$ for any initial condition. Motivated by (3.1) we choose the Lyapunov function candidate

$$V = \tfrac{1}{2}M(\theta)(\dot{\theta} - \dot{\theta}_d, \dot{\theta} - \dot{\theta}_d) + \tfrac{1}{2}K_p(\theta - \theta_d, \theta - \theta_d) \tag{4.1}$$

where K_p satisfies Assumption 3. Differentiating with respect to time yields

$$\dot{V} = M(\theta)(\ddot{\theta} - \ddot{\theta}_d, \dot{\theta} - \dot{\theta}_d) + \tfrac{1}{2}D_\theta M(\theta)(\dot{\theta} - \dot{\theta}_d, \dot{\theta} - \dot{\theta}_d)(\dot{\theta}) + K_p(\dot{\theta} - \dot{\theta}_d, \theta - \theta_d) \tag{4.2}$$

Substituting the dynamics (2.8) for $M(\theta)(\ddot{\theta}, \cdot)$, and exploiting the multilinearity of $D_\theta M(\theta)$ and the skew-symmetry of $A_{23}D_\theta M(\theta)$, we have

$$\begin{aligned}\dot{V} = F(\dot{\theta} - \dot{\theta}_d) &- \tfrac{1}{2}D_\theta M(\theta)(\dot{\theta}_d, \dot{\theta} - \dot{\theta}_d)(\dot{\theta}) + A_{23}D_\theta M(\theta)(\dot{\theta}, \dot{\theta}_d)(\dot{\theta} - \dot{\theta}_d) \\ &- D_\theta G(\theta)(\dot{\theta} - \dot{\theta}_d) - M(\theta)(\ddot{\theta}_d, \dot{\theta} - \dot{\theta}_d) + K_p(\theta - \theta_d, \dot{\theta} - \dot{\theta}_d)\end{aligned} \tag{4.3}$$

Choose F according to

$$\begin{aligned}F(\cdot) = M(\theta)(\ddot{\theta}_d, \cdot) &- K_p(\theta - \theta_d, \cdot) - K_v(\dot{\theta} - \dot{\theta}_d, \cdot) \\ &+ D_\theta G(\theta)(\cdot) + \tfrac{1}{2}D_\theta M(\theta)(\dot{\theta}_d, \cdot)(\dot{\theta}) - A_{23}D_\theta M(\theta)(\dot{\theta}, \dot{\theta}_d)(\cdot)\end{aligned} \tag{4.4}$$

where K_v is positive definite. This is our control law as shown in Fig. 1. Note the simple decomposition into an inner PD loop and the outer dynamic compensation loop. We divide the dynamic compensation into inertial forces and those forces which are quadratic in velocities. We call the quadratic terms 'quasi-Coriolis' forces. With this choice of F

$$\dot{V} = -K_v(\dot{\theta} - \dot{\theta}_d, \dot{\theta} - \dot{\theta}_d) \tag{4.5}$$

which is negative semi-definite in the tracking error. In the position control case we applied LaSalle's theorem at this point. However, in this case, the error dynamics are governed by the following non-autonomous differential equation derived from (4.4) and (2.8).

$$\begin{aligned}M(\theta)(\ddot{\theta} - \ddot{\theta}_d, \cdot) = &-K_p(\theta - \theta_d, \cdot) - K_v(\dot{\theta} - \dot{\theta}_d, \cdot) \\ &- \tfrac{1}{2}D_\theta M(\theta)(\dot{\theta} - \dot{\theta}_d, \cdot)(\dot{\theta}) + A_{23}D_\theta M(\theta)(\dot{\theta}, \dot{\theta} - \dot{\theta}_d)(\cdot)\end{aligned} \tag{4.6}$$

Observe that this differential equation in the tracking error is non-autonomous since it depends on time through $\theta(t)$ and $\dot{\theta}(t)$. Thus, instead of applying LaSalle's theorem we use a theorem of Matrosov to show stability. The technical details are contained in the Appendix, but the key fact that we must check is that $V^{(3)}$ is negative definite when restricted to $E \triangleq \{[(\theta-\theta_d)^T \quad (\dot{\theta}-\dot{\theta}_d)^T]^T \mid \dot{V}=0\}$. Differentiating (4.5) twice and using (4.6) to calculate $(\ddot{\theta}-\ddot{\theta}_d)$ we have that

$$\left.\begin{aligned} V^{(3)} = -2K_v([M(\theta)]^{-1}[K_v](\theta-\theta_d), [M(\theta)]^{-1}[K_v](\theta-\theta_d)) \\ \forall[(\theta-\theta_d)^T, (\dot{\theta}-\dot{\theta}_d)^T]^T \in E \end{aligned}\right\} \tag{4.7}$$

which is a negative definite function of the tracking error. This method of analysis is quite convenient; when $\dot{V}$ goes to zero we simply look at a higher-order derivative! From the Appendix we conclude that the control law described by (4.4) and shown in Fig. 1 gives the desired performance; the tracking error tends to zero. We emphasize the simple structure of the controller given by (4.4). If we choose the K_v and K_p diagonal, then the PD feedback can be done at the joint level in a decoupled manner.

5. Simulation

The control law (4.4) together with the manipulator dynamics (2.8) was simulated using Matrixx (Matrixx is a registered trademark of Integrated Systems Inc.) for a two-link manipulator to verify experimentally the stability of the control law. In addition, we can get a feel for properties of the control scheme which were not assessed in the stability analysis; for example, convergence rates and at which velocities the quasi-Coriolis forces become significant.

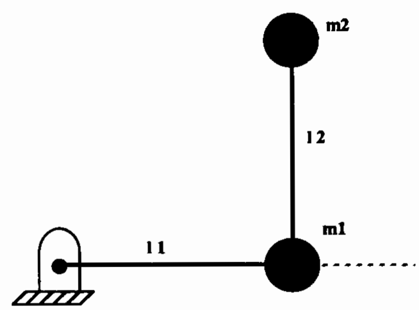

Figure 4. Zero configuration of the robot.

The two-link manipulator used in the simulation is depicted in Fig. 4 with $\theta_1 = 0$ and $\theta_2 = \pi/2$ for the configuration shown. The links have lengths of one, masses of one concentrated on their distal ends, and gravity is taken to be zero. We choose the desired trajectory for θ_2 to be a cosine curve centred at $\pi/2$ to get large variations in the inertia matrix. Also, the desired trajectory for θ_1 is chosen to be a sinusoid of a different frequency so that we can distinguish, to some degree, the effects of the two joint motions on the plots generated by the simulation. The data used for the simulation are the following.

Manipulator dynamics:

$$\left.\begin{aligned} F_1 &= (2\cos\theta_2 + 3)\ddot{\theta}_1 + (1+\cos\theta_2)\ddot{\theta}_2 - \dot{\theta}_2^2 \sin\theta_2 - 2\dot{\theta}_1\dot{\theta}_2 \sin\theta_2 \\ F_2 &= (1+\cos\theta_2)\ddot{\theta}_1 + \ddot{\theta}_2 + \dot{\theta}_1^2 \sin\theta_2 \end{aligned}\right\} \tag{5.1}$$

Control law (from (4.4)):

$$\left.\begin{aligned} F_1 &= (2\cos\theta_2 + 3)\ddot{\theta}_{1d} + (1 + \cos\theta_2)\ddot{\theta}_{2d} - K_{p11}(\theta_1 - \theta_{1d}) \\ &\quad - K_{v11}(\dot{\theta}_2 - \dot{\theta}_{2d}) - (\sin\theta_2)(\dot{\theta}_{1d}\dot{\theta}_2 + \dot{\theta}_1\dot{\theta}_{2d} + \dot{\theta}_2\dot{\theta}_{2d}) \\ F_2 &= (1 + \cos\theta_2)\ddot{\theta}_{1d} + \ddot{\theta}_{2d} - K_{p22}(\theta_2 - \theta_{2d}) - K_{v2}(\dot{\theta}_2 - \dot{\theta}_{2d}) + \sin(\theta_2)\dot{\theta}_1\dot{\theta}_{1d} \end{aligned}\right\} \quad (5.2)$$

PD gains (chosen empirically):

$$K_p = \mathrm{diag}(12{\cdot}0,\ 12{\cdot}0) \quad (5.3)$$

$$K_v = \mathrm{diag}(7{\cdot}0,\ 7{\cdot}0) \quad (5.4)$$

Desired trajectory:

$$\begin{aligned} \theta_{1d} &= -(\pi/2)\cos(0{\cdot}2\pi t) \\ \theta_{2d} &= -(\pi/2)\cos(0{\cdot}4\pi t) \end{aligned} \quad (5.5)$$

The results of the simulation are shown in Figs. 5–9. Figures 5, 6 and 7 verify the convergence of the scheme and Figs. 8 and 9 allow us to compare the relative sizes of the quasi-Coriolis terms and the total control force. Figures 10 and 11 show the performance of the inner PD loop without the dynamic compensation.

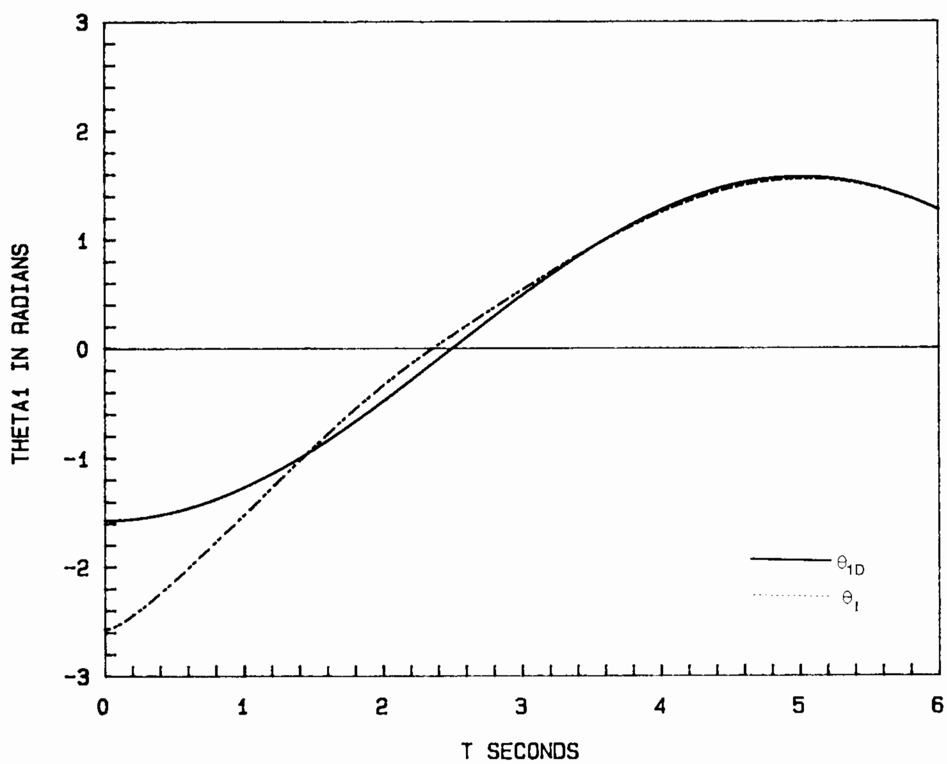

Figure 5. PD+ controller performance: joint 1 tracking.

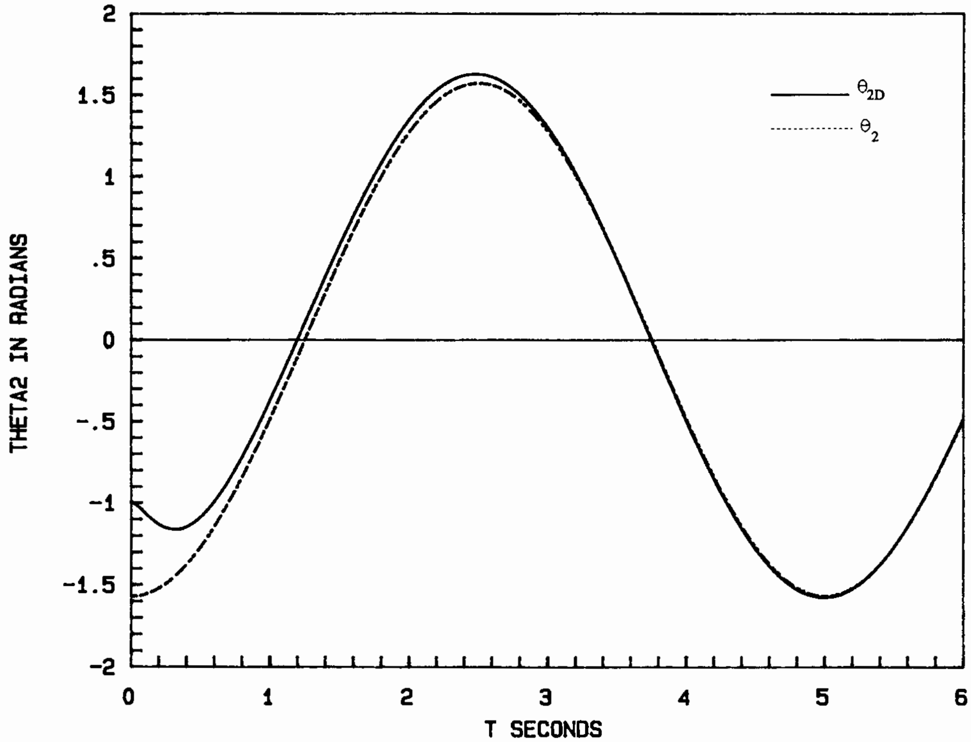

Figure 6. PD+ controller performance: joint 2 tracking.

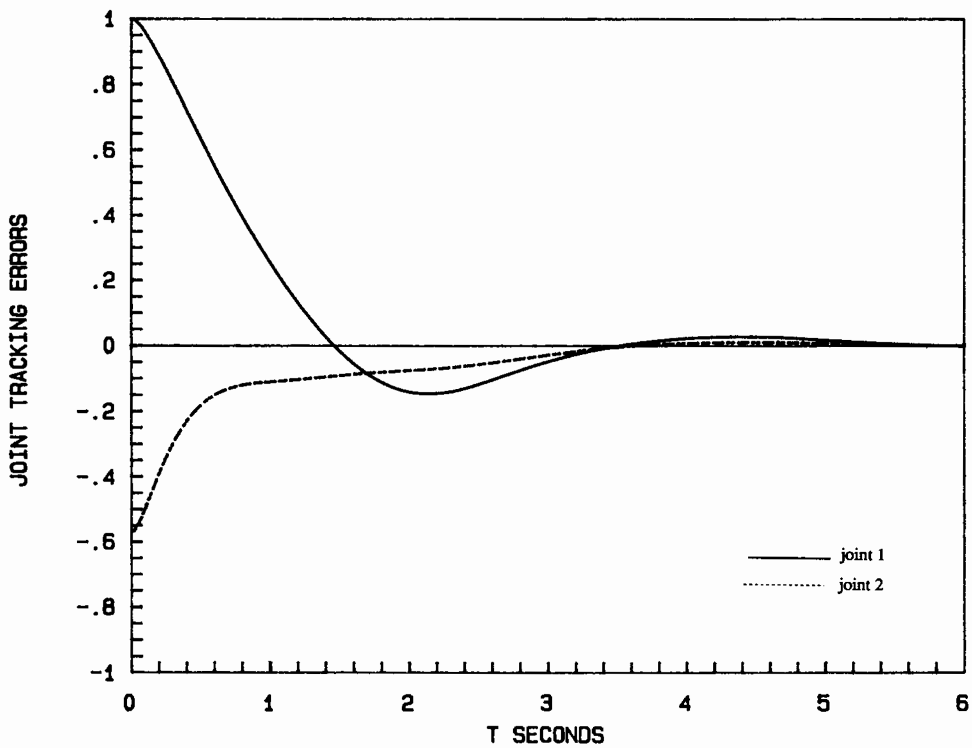

Figure 7. PD+ controller performance: joint tracking errors.

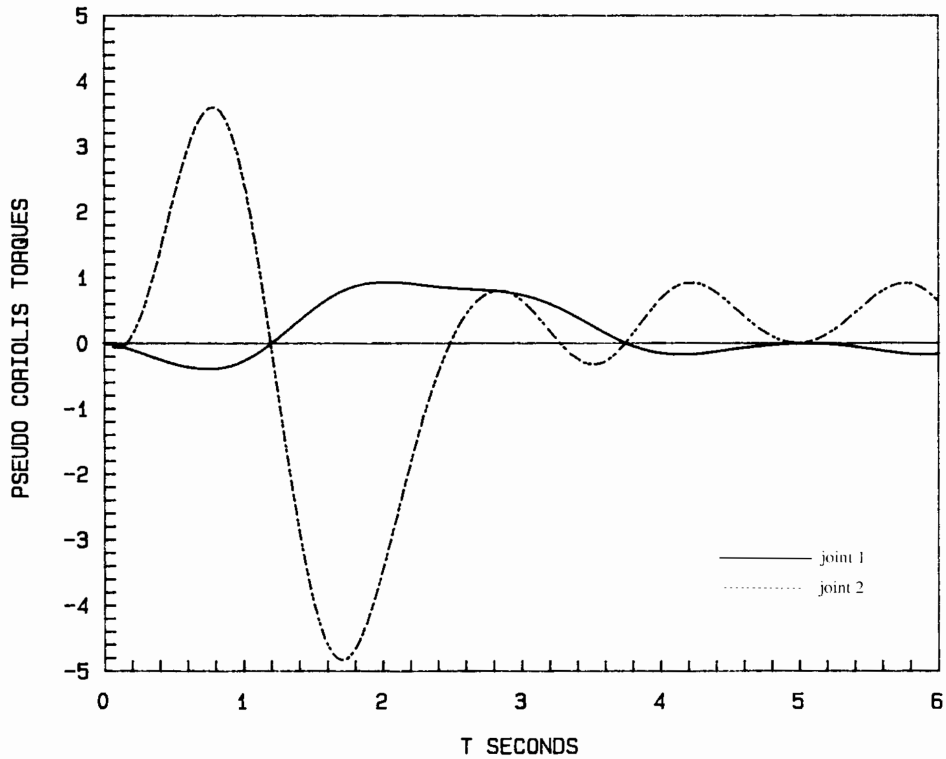

Figure 8. PD+ controller pseudo joint torques.

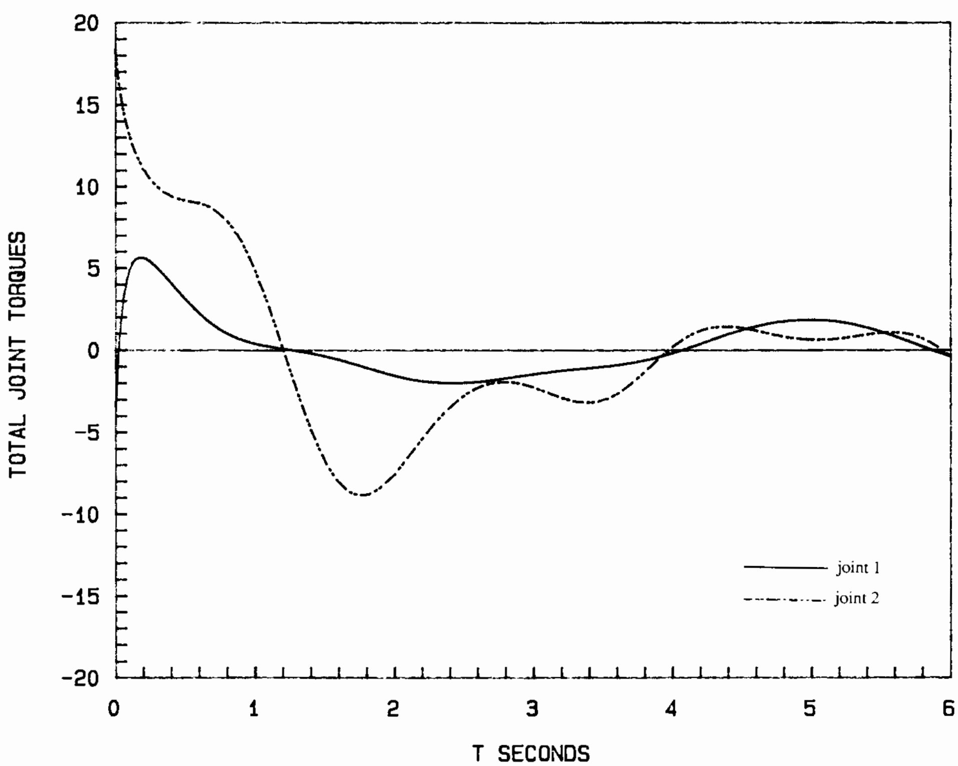

Figure 9. PD+ controller joint torques.

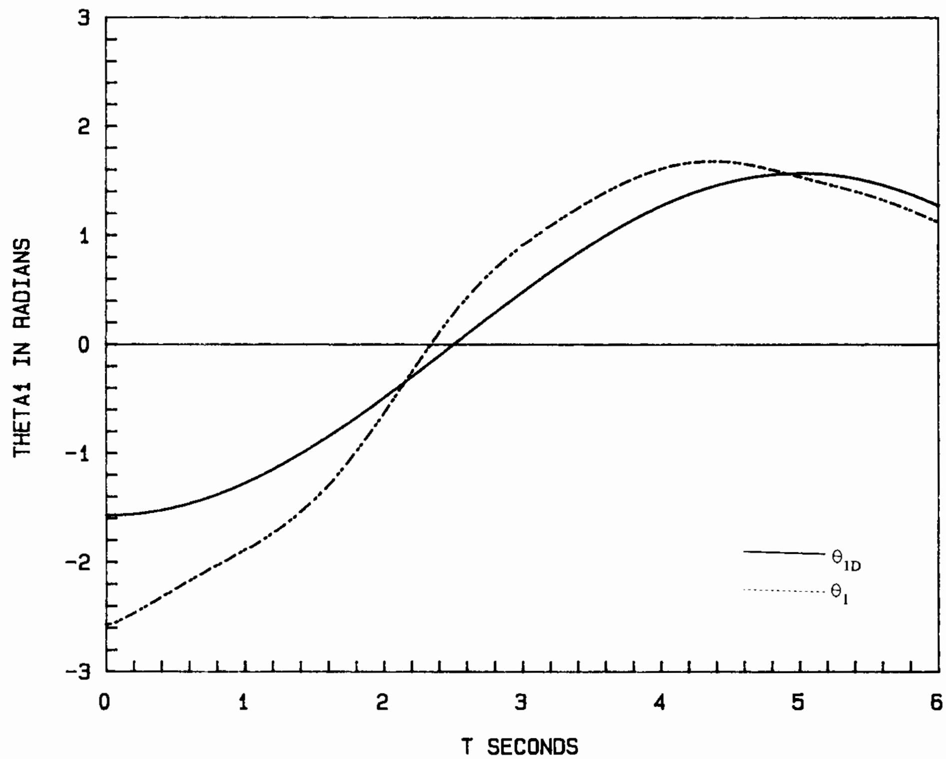

Figure 10. Uncompensated PD controller performance: joint 1 tracking.

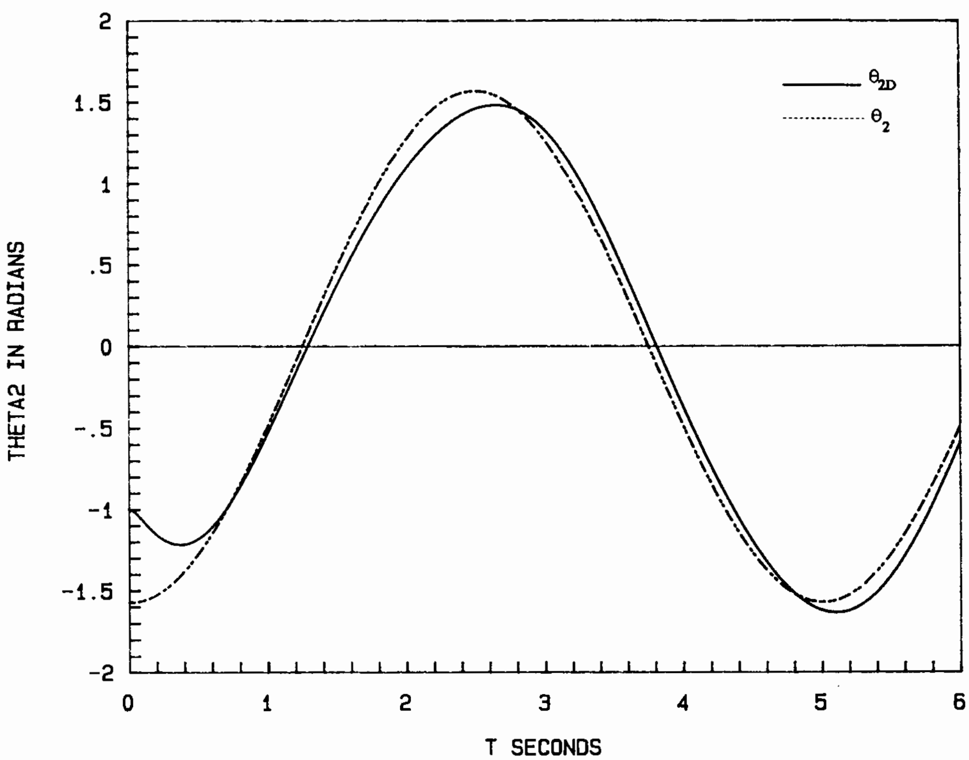

Figure 11. Uncompensated PD controller performance: joint 2 tracking.

6. Conclusion

The PD+ tracking controller developed here and the computed torque controller are two basic globally stable tracking controllers. The computed torque scheme is the obvious *linearizing* controller and our tracking scheme is a simple PD controller with dynamic compensation. The feedback is linear in our scheme with compensation in the form of quasi-Coriolis forces, gravity forces, and feed-forward of the inertial forces. Moreover, the tracking controller is the natural extension of Takegaki and Arimoto's scheme to the tracking case.

It is important to note that we can take advantage of the hamiltonian structure of the unforced robot dynamics to build our Lyapunov functions. In this sense the control law is a natural one. However, questions remain as to whether we should use this scheme in the place of the computed torque scheme. The computed torque scheme has the property that the error dynamics are independent of the configuration and therefore the inertia matrix. Intuitively, we expect that the error dynamics should 'slow down' when the manipulator is in an extended position and the inertia matrix is, roughly speaking, larger in some directions. The controller we have presented has this property as the feedback gains are independent of the configuration. Thus, when the manipulator arm is extended, the error dynamics will be slower. The error dynamics slow down as the inertia increases in the sense that the poles of the linearization move away from the origin. However, the damping ratio of these poles also varies. This is worthy of consideration and is a topic for further research.

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Appendix

Stability analysis

Here we give the formal proof that the control law given by (4.4) guarantees that $[0 \quad 0]^T$ is a globally asymptotically stable equilibrium point of the tracking error. We first define functions of 'class K'.

Definition

A continuous function $\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class K if: (1) $\alpha(p)$ is strictly increasing; (2) $\alpha(0) = 0$.

Our result relies on the following theorem of Matrosov concerning the differential equation $\dot{x} = f(t, x)$ with an equilibrium point at 0 at time t_0. Let $\dot{V}(t, x)$ and $\dot{W}(t, x)$ denote derivatives along solution trajectories of the function $V(t, x)$ and $W(t, x)$. Then we can state the following.

Theorem 1 (Matrosov)

Let $\Omega \in \mathbb{R}^n$ be an open connected region in $\mathbb{R}^n$ containing the origin. If there exist two C^1 functions $V: [t_0, \infty) \times \Omega \rightarrow \mathbb{R}$, $W: [t_0, \infty) \times \Omega \rightarrow \mathbb{R}$; a C^0 function $V^*: \Omega \rightarrow \mathbb{R}$; three functions a, b, c of class K such that, for every $(x, t) \in [t_0, \infty) \times \Omega$,

(1) $a(\|x\|) \leqslant V(t, x) \leqslant b(\|x\|)$

(2) $\dot{V}(t, x) \leqslant V^*(x) \leqslant 0$. Define $E \triangleq \{x \in \Omega \mid V^*(x) = 0\}$.

(3) $|W(t, x)|$ is bounded

(4) $\max(d(x, E), |\dot{W}(t, x)|) \geqslant c(\|x\|)$

(5) $\|f(t, x)\|$ is bounded.

Choosing $\alpha > 0$ such that $\bar{B}_\alpha \subset \Omega$, define for all $t \in [t_0, \infty)$

$$V_{t,\alpha}^{-1} = \{x \in \Omega : V(t, x) \leqslant a(\alpha)\}$$

Then

(*a*) For all $x_0 \in V_{t_0,\alpha}^{-1}$, $x(t)$ tends to zero uniformly in t_0, x_0 as t tends to infinity.

(*b*) The origin is uniformly asymptotically stable.

Proof

See Rouche *et al.* (1977).

Remarks

Condition (1) is no surprise. This is a common requirement in Lyapunov theorems needed to guarantee uniform stability. Condition (2) is a fairly weak requirement considering the fact that we get asymptotic stability out of this theorem. A requirement that $\dot{V}$ be negative definite is usual (see Vidyasagar 1978). The region E defined in condition (2) is the problem area. The only point in E where we want to get stuck is the origin; however, the rate of change of the potential is zero in all of E. To compensate the weak condition (2), conditions (3) and (4) are needed. These tell us, roughly speaking, that near E and away from the origin, the rate of change of a second *bounded* auxiliary function is of constant sign and bounded away from zero. Thus, the state cannot remain near E and away from the origin without driving W beyond its bounds (i.e. the only place in E to which the state can converge is the origin).

Condition (4) of the theorem is easier for us to verify if we use the following lemma.

Lemma

Condition (4) of the theorem is satisfied if conditions (4)′ below are satisfied.

4′(*a*) $\dot{W}(x, t)$ is continuous in both arguments and depends on time in the following way. $\dot{W}(x, t) = g(x, \beta(t))$ where g is continuous in both of its arguments. $\beta(t)$ is also continuous and its image lies in a bounded set K_1. (For simplicity, we simply say that $\dot{W}(x, t)$ *depends on time continuously through a bounded function.*)

4′(*b*) There exists a class K function, k, such that $|\dot{W}(x, t)| \geqslant k(\|x\|) \forall x \in E$ and $t \geqslant t_0$.

Proof

Since $\beta(t) \subset K_1$ and $\dot{W}$ is continuous in β for all $t \geqslant t_0$ we have that $Y(x) \triangleq \min_{\beta \in K_1} |g(x, \beta)|$ exists and is continuous. Now consider

$$c_1(\alpha) \triangleq \min_{\eta \in \Omega, \|\eta\| \geqslant \alpha} \max(d(\eta, E), |Y(\eta)|) \tag{A 1}$$

which is defined for all $\alpha = \|x\|$, x in Ω. The set over which the minimization occurs is decreasing with increasing α, so $c_1(\alpha)$ is non-decreasing with respect to α. Observe that

$c_1(\alpha) \geqslant 0$ and that $c_1(\alpha) = 0$ implies that the minimum in (A 1) is attained for $\eta = 0$ by condition $4'(b)$. Thus, in addition to being non-decreasing, $c_1(0) = 0$, and $c_1(p) > 0$ for all $p > 0$. To show condition (4) define the *strictly* increasing function $c(\cdot)$ by

$$c(p) \triangleq \int_0^p c_1(\sigma) \exp[-(p-\sigma)]\, d\sigma \tag{A 2}$$

It follows from the properties of c_1 that $c(\alpha) \leqslant c_1(\alpha)$ and $c(p)$ satisfies the conditions of the definition of a function of class K. Thus, condition (4) holds:

$$\max(d(x, E), |\dot{W}(t, x)|) \geqslant c(\|x\|) \tag{A 3}$$

□

This lemma and Theorem 1 are now applied to show stability.

Theorem 2

Suppose that Assumptions 1–5 are satisfied. Then, for the dynamical system described by (2.8) and (4.4), (0, 0) is a globally asymptotically stable equilibrium point of the tracking error $x \triangleq [\theta - \theta_d \quad \dot{\theta} - \dot{\theta}_d]^T$.

Proof

We simply check the conditions of the theorem for $V = 1/2M(\theta)(\dot{\theta} - \dot{\theta}_d, \dot{\theta} - \dot{\theta}_d) + 1/2K_p(\theta - \theta_d, \theta - \theta_d)$ and $\dot{x} = f(x, t)$ defined implicitly by (4.6), where we are taking x to be $[\theta - \theta_d \quad \dot{\theta} - \dot{\theta}_d]^T$. Also, to show that the stability is global, we must show that we can choose α so that $V_{t_0,\alpha}^{-1}$ contains an arbitrary initial tracking error.

(1) Clearly, by the assumptions (1 and 3) on $M(\theta)$ and K_p, condition (1) is satisfied for $(t, x) \in [t_0, \infty) \times \mathbb{R}^{2m}$. In fact, we can choose $a(\cdot)$ such that $a(p) \to \infty$ as $p \to \infty$. (Note that Ω can be chosen arbitrarily large here. It will be restricted when condition (5) is checked.)

(2) From (4.5), $\dot{V} = -K_v(\dot{\theta} - \dot{\theta}_d, \dot{\theta} - \dot{\theta}_d) \leqslant 0$ and depends only on x so condition (2) holds for $(t, x) \in [t_0, \infty) \times \mathbb{R}^{2m}$.

Before we proceed, we establish the boundedness and continuity of several functions. Conditions (1) and (2) of Matrosov's theorem are satisfied and therefore $\theta - \theta_d$ and $\dot{\theta} - \dot{\theta}_d$ are bounded. Moreover, θ_d and $\dot{\theta}_d$ are bounded and so θ and $\dot{\theta}$ are bounded as well. Since $M(\theta)$ is twice continuously differentiable and θ is bounded, the tensors $D_\theta M(\theta)$ and $D_\theta^2 M(\theta)$ are bounded. Furthermore, $\ddot{\theta} - \ddot{\theta}_d$ is continuous in the tracking error and depends continuously on time through θ and $\dot{\theta}$ which are bounded (this is important for verifying $4'(a)$).

With these bounds in mind, it is easy to see that the RHS of (4.6) is bounded. Recall that the minimum singular value of $M(\theta)$ is bounded away from zero so we have $(\ddot{\theta} - \ddot{\theta}_d)$ is bounded.

Finally, observe that the solution to the differential equation obtained by combining (2.8) and (4.4) is continuously differentiable and so θ, $\dot{\theta}$ and $\ddot{\theta}$ are continuous.

We proceed with checking condition (3).

(3) Define $W(x, t) \triangleq \ddot{V}(x, t)$. We have from (4.5) that

$$\ddot{V} = 2K_v(\dot{\theta} - \dot{\theta}_d, \ddot{\theta} - \ddot{\theta}_d)\, \forall x \in E \tag{A 4}$$

It follows from the bounds above that this is bounded.

(4) By the lemma it suffices to verify $4'(a)$ and $4'(b)$. Computing $\dot{W} = V^{(3)}$ we have

$$\dot{W} = 2K_v(\ddot{\theta} - \ddot{\theta}_d, \ddot{\theta} - \ddot{\theta}_d) + 2K_v\left(\dot{\theta} - \dot{\theta}_d, \frac{d}{dt}(\ddot{\theta} - \ddot{\theta}_d)\right) \tag{A 5}$$

All arguments, except for $\frac{d}{dt}(\ddot{\theta} - \ddot{\theta}_d)$, in (A 5) are known (at this point in the proof) to be continuous in the tracking error and depend continuously on time through a bounded function. (We do not write $\theta_d^{(3)}$ or $\theta_d^{(3)}$ since they may not exist.) To establish $4'(a)$ we show that $\frac{d}{dt}(\ddot{\theta} - \ddot{\theta}_d)$ is continuous with respect to the tracking error and continuous with respect to time through a bounded function. Differentiating (4.6) with respect to time we have

$$\begin{aligned} M(\theta)\left(\frac{d}{dt}(\ddot{\theta} - \ddot{\theta}_d), \cdot\right) &+ D_\theta M(\theta)(\ddot{\theta} - \ddot{\theta}_d, \cdot)(\dot{\theta}) \\ &= -K_p(\dot{\theta} - \dot{\theta}_d, \cdot) - K_v(\ddot{\theta} - \ddot{\theta}_d, \cdot) \\ &\quad - \frac{1}{2} D_\theta^2 M(\theta)(\dot{\theta} - \dot{\theta}_d, \cdot)(\dot{\theta})(\dot{\theta}) - \frac{1}{2} D_\theta M(\theta)(\ddot{\theta} - \ddot{\theta}_d, \cdot)(\dot{\theta}) \\ &\quad - \frac{1}{2} D_\theta M(\theta)(\dot{\theta} - \dot{\theta}_d, \cdot)(\ddot{\theta}) + A_{23} D_\theta M(\theta)(\ddot{\theta}, \dot{\theta} - \dot{\theta}_d)(\cdot) \\ &\quad + A_{23} D_\theta M(\theta)(\dot{\theta}, \ddot{\theta} - \ddot{\theta}_d)(\cdot) + A_{23} D_\theta^2 M(\theta)(\dot{\theta}, \dot{\theta}_d)(\cdot)(\dot{\theta}) \end{aligned} \tag{A 6}$$

All terms in the RHS of (A 6) are continuous with respect to the tracking error and depend on time continuously through a bounded function. In addition, the minimum singular value of $M(\theta)$ is bounded away from zero. Hence, $\frac{d}{dt}(\ddot{\theta} - \ddot{\theta})$ is continuous in the tracking error and depends on time continuously through a bounded function and $4'(a)$ is established for $\Omega = \mathbb{R}^{2m}$. Condition $4'(b)$ follows from (4.7) and the fact that the maximum singular value of $M(\theta)$ is bounded.

(5) Since $\theta(t)$ and $\dot{\theta}(t)$ are bounded, and the RHS of (4.6) is continuous in these variables and the trajectory error, the RHS of (4.6) is bounded for all $(x, t) \in \Omega \times \mathbb{R}_+$ when Ω is bounded. Moreover, $[M(\theta)]$ is bounded and so $\| f(t, x) \|$, defined implicitly by (4.6), is bounded on $[0, \infty) \times \Omega$ when Ω is chosen bounded and the condition is satisfied.

Thus the conditions of the theorem are satisfied. Moreover, for arbitrary initial conditions, we can use condition (1) to find an α and Ω such that $x_0 \in V_{t_0,\alpha}^{-1}$. Thus, by Matrosov's theorem, the origin is a globally asymptotically stable equilibrium point of the tracking error. □

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