

## Cubic Splines on Curved Spaces

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We consider a second-order problem in the calculus of variations, with an application to robotics in mind. The analysis is carried out on a general Riemannian manifold  $M$  and then specialized to the case where  $M$  is the Lie group  $SO(3)$  of rotations in  $\mathbb{R}^3$ . For  $SO(3)$ , the Euler–Lagrange equations reduce to interesting nonlinear systems of ordinary differential equations in  $\mathbb{R}^3$ .

### 1. Introduction

LET  $R$  be a reference point on a rigid body moving in  $\mathbb{R}^3$ . Let  $(x_R, y_R, z_R)$  be an ordered orthonormal set of vectors through  $R$ , fixed relative to the body. Then, at any time  $t$ , the configuration of the body is given by the ordered pair  $(R(t), x(t)) \in \mathbb{R}^3 \times SO(3)$ . Here  $x(t)$  is the orthogonal transformation that transforms  $(x_R(0), y_R(0), z_R(0))$  to  $(x_R(t), y_R(t), z_R(t))$ . It is sometimes the case that points  $(R(t_k), x(t_k))$  are specified for various times  $t_k$ , and a curve interpolating these points is required. This happens, for example, in robotics and the present paper is written with this application in mind.

We may interpolate the points  $R(t_k)$  in  $\mathbb{R}^3$  in various ways, independently of the  $x(t_k)$ ; there remains the problem of interpolating the  $x(t_k)$ . It might seem natural to join each  $x(t_{k-1})$  to  $x(t_k)$  by a geodesic, but an interpolation defined in this way will usually fail to be differentiable at the junction points  $t_k$ , and instantaneous changes in momentum are undesirable from an engineering point of view. Alternatively, we might cover  $SO(3)$  by coordinate charts and essentially reduce to interpolation in  $\mathbb{R}^3$ ; it is recognised by engineers that this is an untidy procedure, because it depends on choices of charts.

Here we consider one of several natural ways of interpolating on  $SO(3)$  using differentiable curves. We suppose that the  $\dot{x}(t_k)$  are prescribed as well as the  $x(t_k)$ . Then, for each  $k$ , a differentiable curve  $\gamma: [t_{k-1}, t_k] \rightarrow SO(3)$  is said to be *feasible* when  $\gamma(t_{k-1})$ ,  $\gamma(t_k)$ ,  $\dot{\gamma}(t_{k-1})$ , and  $\dot{\gamma}(t_k)$  take on the prescribed values: a track sum of feasible curves is once differentiable, but usually not twice. Without loss, let  $t_{k-1} = 0$  and  $t_k = 1$ .

Since there are many feasible curves, we take the opportunity to select curves that have an additional desirable property. We require that, among all feasible curves  $\gamma$ , the choice  $\gamma = x$  should minimize the quantity

$$\Phi(\gamma) = \int_0^1 \langle \nabla_{d/dt} \dot{\gamma}, \nabla_{d/dt} \dot{\gamma} \rangle dt$$

or, since we consider only first-order variations of  $x$ , the functional  $\Phi$  should at least be stationary at  $x$ . Here  $\langle \cdot, \cdot \rangle$ , denotes a natural way of taking inner products of vectors tangential to  $SO(3)$ , namely a left-invariant and right-invariant Riemannian metric. Also  $\nabla$  is the symmetric covariant derivative compatible with  $\langle \cdot, \cdot \rangle$ , namely the Levi-Civita affine connection.

Engineering considerations suggest that we minimize  $\Phi(\gamma)$ , which is the average of the squared norm of the angular acceleration. Geodesics are unlikely to be feasible, but if there happens to be a feasible geodesic  $z$  then  $\gamma = z$  minimizes  $\Phi$  because the integrand vanishes. Our problem can be formulated for any Riemannian manifold  $M$  and not just for  $SO(3)$ . When  $M = \mathbb{R}$ , and we interpolate in the way described above, we obtain a cubic spline of type I, according to Ahlberg *et al.* (1967: Thm. 3.4.3).

In Section 2 we make some definitions which may be difficult to digest but are needed to compute the first variation of  $\Phi$ : we need to differentiate covariantly cross-sections of a bundle induced from  $TM$ . In Section 3 we state and prove a necessary condition for  $x$  to minimize  $\Phi$ ; this is carried out for a general Riemannian manifold  $M$ . Bearing in mind the situation for cubic splines on  $\mathbb{R}$ , especially Ahlberg *et al.* (1967: Thm 3.4.3), it is unsurprising that  $\nabla_{\partial/\partial t}^3 \dot{x}$  appears in the Euler-Lagrange equations, but there is also a term involving the Riemannian curvature tensor; to account for this, we require a careful analysis of the geometry of  $SO(3)$ .

In Section 4 we specialize to the example discussed first, namely  $M = SO(3)$ . We are led to a 3-parameter family of systems (4.2) of three second-order nonlinear autonomous ordinary differential equations.

## 2. Preliminaries

Let  $V$  be a smooth real vector bundle over a smooth  $m$ -manifold  $M$ , where  $M$  may possibly have a boundary. Let  $B$  be a smooth manifold, possibly with boundary, and let  $\nu: B \rightarrow M$  be a smooth map. Let  $E$  be the real vector bundle  $\nu^*V$  over  $B$  induced from  $V$  by  $\nu$ . Then we have a pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{\pi} & V \\ \downarrow & \nu & \downarrow \\ B & \longrightarrow & M \end{array}$$

Liftings of  $\nu$  to  $V$  correspond naturally to cross-sections of  $E$ .

In particular if  $s$  is a smooth cross-section of  $V$  then  $s \circ v$  is a lifting of  $v$  to  $V$ . The corresponding cross-section of  $E$  is denoted by  $v^*s$  and called the cross-section *induced from  $s$  by  $v$* . Consider the module  $\Gamma(E)$  of all cross-sections of  $E$  over the ring  $C^\infty(B; \mathbb{R})$  of smooth real-valued functions on  $B$ . Composition with  $v$  defines a ring homomorphism  $v^* : C^\infty(M; \mathbb{R}) \rightarrow C^\infty(B; \mathbb{R})$ , and induction by  $v$  defines a module homomorphism  $\Gamma(V) \rightarrow \Gamma(E)$  over  $v^*$  which embeds  $\Gamma(V)$  as the  $C^\infty(M; \mathbb{R})$ -module of cross-sections induced by  $v$ . Then, locally at least,  $\Gamma(E)$  is generated as a  $C^\infty(B; \mathbb{R})$ -module by  $\Gamma(V)$  embedded in this way.

Let  $\nabla$  be a covariant derivative for the vector bundle  $V$ . Let  $X' \in (TB)_{x'}$ , where  $x' \in B$ , and let  $s'$  be a cross-section of  $E$ . In a neighbourhood of  $x'$ , we write

$$s' = f_i \cdot v^*s_i, \tag{1}$$

where  $f_i \in C^\infty(B; \mathbb{R})$ ,  $s_i \in \Gamma(V)$ , and summation over a finite index set is understood. We define  $\nabla'_{X'}s'$  to be

$$f_i(x') \cdot v^*(\nabla_{d_{v_x}(X')}s_i) + X'(f_i) \cdot v^*s_i.$$

Different choices of the  $f_i$  and  $v_i$  do not affect  $\nabla'_{X'}s'$ , so long as (1) is satisfied: it is sufficient to verify this when  $s'$  is itself induced by  $v$ . Then  $\nabla'$  is a covariant derivative for the bundle  $E$ , and we call it the covariant derivative *induced from  $\nabla$* . Induction is functorial, in the following sense. Let  $w : C \rightarrow B$  be a second smooth map of manifolds with boundary, and let  $\nabla''$  be the covariant derivative induced from  $\nabla'$  by  $w$  on  $F = w^*E$ . Then  $\nabla''$  is also the covariant derivative on  $F \cong (v \circ w)^*V$  induced from  $\nabla$  by  $v \circ w$ .

Let  $\lambda : B \rightarrow V$  be a smooth lifting of  $v$ . Then  $\lambda$  corresponds naturally to a cross-section  $s'$  of  $E$ . Given a vector field  $X'$  on  $B$ , we form the cross-section  $\nabla'_{X'}s'$  of  $E$  over  $x'$ , and the corresponding lifting of  $v$  to  $V$  is denoted by  $\nabla_{X'}\lambda$ . Given  $w : C \rightarrow B$  and  $X'' \in (TC)_{x''}$ , we can form the lifting  $\lambda \circ w$  of  $v \circ w$  to  $V$ . Then  $\nabla_{X''}(\lambda \circ w)$  is a lifting of  $v \circ w$  to  $V$  and, because induction is functorial, we have

$$(\nabla_{X''}(\lambda \circ w))_{x''} = (\nabla_{X'}(\lambda))_{w(x'')}$$

when  $X'_{w(x'')} = dw_{x''}(X''_{x''})$ .

*Remarks.* (1) The use of the symbol  $\nabla$  in this way requires special care. For example, it may be the case that  $v$  is constant on  $w(C)$ , where  $C = \mathbb{R}$ , so that  $d(v \circ w)/dt = 0$ ; it does not follow that  $\nabla_{\partial/\partial t}(\pi \circ \lambda) = 0$ .

(2) In Section 3 we have  $v : (-\epsilon, \epsilon) \times [0, 1] \rightarrow M$ , and various liftings  $\lambda$  of  $v$  to  $TM$ ; for example  $\lambda$  might be  $\dot{x}$ . Now  $\partial/\partial t$  is a vector field defined on  $(-\epsilon, \epsilon) \times [0, 1]$ , and  $\nabla_{\partial/\partial t}\lambda$  is a lifting of  $v$  to  $TM$ . If  $w$  is the inclusion of  $[0, 1] \cong \{0\} \times [0, 1]$  in  $(-\epsilon, \epsilon) \times [0, 1]$ , then

$$(\nabla'_{d/dt}\lambda \circ w)_t = (\nabla_{\partial/\partial t}\lambda)_{(0,t)}$$

and so  $(\nabla_{\partial/\partial t}\lambda)_{(0,t)}$  does not depend on the values of  $\lambda$  at points  $(h, t)$  where  $h \neq 0$ . It follows that the expression in the Theorem of Section 3 depends on  $x$  alone, and not on the variation  $v'$  used to define  $v$ .

Suppose now that  $\nabla$  is compatible with a fibrewise metric  $\langle \cdot, \cdot \rangle$  on  $V$ . Let  $\langle \cdot, \cdot \rangle'$  be the induced metric on  $E$ . Then  $\nabla'$  is compatible with  $\langle \cdot, \cdot \rangle'$ .

### 3. Calculus of variations on $M$

Let  $M$  be a Riemannian manifold of dimension  $m \geq 2$ . We have in mind  $M = \text{SO}(3)$ , but it is helpful to consider first what happens when there are no geometric subtleties. When  $M = \mathbb{R}$ , the functional  $\Phi$  involves second-order derivatives and its critical points can be determined in a standard way, as in Gelfand & Fomin (1963: Ch. 2 § 11). Here we extend the classical argument, namely we integrate by parts twice to transform  $[\partial(\Phi(v'(h))/\partial h)]_{h=0}$ , but the geometry of  $M$  enters in an essential and nontrivial way.

Fix  $y_0, y_1 \in M$ ,  $v_0 \in (TM)_{y_0}$ , and  $v_1 \in (TM)_{y_1}$ . Consider the space  $F_{v_0, v_1}$  of smooth maps  $\gamma: [0, 1] \rightarrow M$  where  $\gamma(0) = y_0$ ,  $\gamma(1) = y_1$ ,  $\dot{\gamma}(0) = v_0$ , and  $\dot{\gamma}(1) = v_1$ . Define  $\Phi: F_{v_0, v_1} \rightarrow \mathbb{R}$  by

$$\Phi(\gamma) = \int_0^1 \langle \nabla_{\partial/\partial t} \dot{\gamma}, \nabla_{\partial/\partial t} \dot{\gamma} \rangle dt,$$

where  $\nabla$  is the symmetric covariant derivative compatible with the Riemannian metric  $\langle \cdot, \cdot \rangle$ .

**THEOREM 1.**  $x \in F_{v_0, v_1}$  is a critical point of  $\Phi$  if and only if  $x$  satisfies

$$\{\nabla_{\partial/\partial t}^3 \dot{x} + R(\nabla_{\partial/\partial t} \dot{x}, \dot{x})(\dot{x})\}_t = 0$$

for all  $t \in [0, 1]$ , where  $R$  is the Riemannian curvature tensor of  $\nabla$ .

*Proof.* Consider a variation  $v': (-\varepsilon, \varepsilon) \rightarrow F_{v_0, v_1}$ , where  $v'(0) = x$ ; let  $v: (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$  be given by  $v(h, t) = v'(h)(t)$ . Let  $E$  be the induced bundle  $v^*(TM)$  over  $B = (-\varepsilon, \varepsilon) \times [0, 1]$ , with the fibrewise metric  $\langle \cdot, \cdot \rangle'$  induced from  $\langle \cdot, \cdot \rangle$ .

Let  $\dot{x}', X' \in \Gamma(E)$  correspond to the liftings  $\dot{x}$  and  $X$  of  $v$  given by

$$(h, t) \mapsto dv_{(h,t)}(\partial/\partial t), \quad (h, t) \mapsto (\nabla_{\partial/\partial t} \dot{x})_{(h,t)}.$$

Then

$$\begin{aligned} \partial/\partial h \langle \nabla_{\partial/\partial t} \dot{x}, \nabla_{\partial/\partial t} \dot{x} \rangle_{(h,t)} &= \nabla'_{\partial/\partial h} \langle X', X' \rangle'_{(h,t)} \\ &= 2 \langle \nabla'_{\partial/\partial h} X', X' \rangle' = 2 \langle \nabla'_{\partial/\partial h} \nabla'_{\partial/\partial t} \dot{x}', X' \rangle'. \end{aligned}$$

Let  $R'$  be the Riemannian curvature tensor of  $\nabla'$ . Since

$$[\partial/\partial h, \partial/\partial t] = 0,$$

we have

$$\nabla'_{\partial/\partial h} \nabla'_{\partial/\partial t} \dot{x}' = \nabla'_{\partial/\partial t} \nabla'_{\partial/\partial h} \dot{x}' + R'(\partial/\partial h, \partial/\partial t)(\dot{x}').$$

The first variation of  $\Phi$  with respect to  $v'$  is  $\partial\Phi(v'(h))/\partial h$ , and

$$\begin{aligned} \frac{1}{2} \frac{\partial\Phi(v'(h))}{\partial h} &= \int_0^1 \langle \nabla'_{\partial/\partial h} \nabla'_{\partial/\partial t} \dot{x}', X' \rangle'_{(h,t)} dt \\ &= \int_0^1 \left\langle \nabla'_{\partial/\partial t} \nabla'_{\partial/\partial h} \dot{x}' + R' \left( \frac{\partial}{\partial h}, \frac{\partial}{\partial t} \right) (\dot{x}'), X' \right\rangle'_{(h,t)} dt. \end{aligned}$$

To integrate by parts, note that

$$\begin{aligned} \partial/\partial t \langle \nabla'_{\partial/\partial h} \dot{X}', X' \rangle &= \nabla'_{\partial/\partial t} \langle \nabla'_{\partial/\partial h} \dot{X}', X' \rangle' \\ &= \langle \nabla'_{\partial/\partial t} \nabla'_{\partial/\partial h} \dot{X}', X' \rangle' + \langle \nabla'_{\partial/\partial h} \dot{X}', \nabla'_{\partial/\partial t} X' \rangle'. \end{aligned}$$

So  $\frac{1}{2} \partial \Phi(v'(h))/\partial h$  becomes

$$\begin{aligned} &[\langle \nabla'_{\partial/\partial h} \dot{X}', X' \rangle'_{(h,t)}]_0^1 - \int_0^1 \langle \nabla'_{\partial/\partial h} \dot{X}', \nabla'_{\partial/\partial t} X' \rangle'_{(h,t)} dt \\ &+ \int_0^1 \left\langle R' \left( \frac{\partial}{\partial h}, \frac{\partial}{\partial t} \right) (\dot{X}'), X' \right\rangle'_{(h,t)} dt, \end{aligned}$$

where  $h$  is fixed.

**SUBLEMMA** Both  $(\nabla'_{\partial/\partial h} \dot{X}')_{(h,0)}$  and  $(\nabla'_{\partial/\partial h} \dot{X}')_{(h,1)}$  are zero.

*Proof of sublemma.* Because  $v'$  is a variation through curves with fixed endpoints,  $v'|_{(-\epsilon, \epsilon) \times \{0\}}$  and  $v'|_{(-\epsilon, \epsilon) \times \{1\}}$  are the constant maps to  $y_0$  and  $y_1$  respectively. We can therefore identify  $E|_{(-\epsilon, \epsilon) \times \{0\}}$  with  $(TM)_{y_0}$  and  $E|_{(-\epsilon, \epsilon) \times \{1\}}$  with  $(TM)_{y_1}$ . Cross-sections of  $E|_{(-\epsilon, \epsilon) \times \{0\}}$  and  $E|_{(-\epsilon, \epsilon) \times \{1\}}$  correspond to maps from  $(-\epsilon, \epsilon)$  into the vector spaces  $(TM)_{y_0}$  and  $(TM)_{y_1}$ , and  $\nabla'_{\partial/\partial h}$  corresponds to  $\partial/\partial h$  in either case.

Because  $v'$  is a variation through curves whose initial and final velocities are fixed,  $\dot{X}'|_{(-\epsilon, \epsilon) \times \{0\}}$  and  $\dot{X}'|_{(-\epsilon, \epsilon) \times \{1\}}$  correspond to the constant maps to  $v_0$  and  $v_1$  respectively. This proves our sublemma.

According to our sublemma, the first term in our expression for  $\frac{1}{2} \partial \Phi(v'(h))/\partial h$  is zero. Consequently

$$\begin{aligned} \frac{1}{2} \frac{\partial \Phi(v'(h))}{\partial h} &= \\ &= - \int_0^1 \langle \nabla'_{\partial/\partial h} \dot{X}', \nabla'_{\partial/\partial t} X' \rangle'_{(h,t)} dt + \int_0^1 \left\langle R' \left( \frac{\partial}{\partial h}, \frac{\partial}{\partial t} \right) (\dot{X}'), X' \right\rangle'_{(h,t)} dt. \end{aligned}$$

Let  $W' \in \Gamma(E)$  correspond to the lifting  $W$  of  $v$  given by

$$(h, t) \mapsto dv_{(h,t)}(\partial/\partial h).$$

Because  $\nabla$  is symmetric and  $[\partial/\partial h, \partial/\partial t] = 0$ , it follows that

$$\nabla'_{\partial/\partial h} \dot{X}' = \nabla'_{\partial/\partial t} W',$$

and thus

$$\frac{1}{2} \frac{\partial \Phi(v'(h))}{\partial h} = - \int_0^1 \langle \nabla'_{\partial/\partial t} W', \nabla'_{\partial/\partial t} X' \rangle'_{(h,t)} dt + \int_0^1 \left\langle R' \left( \frac{\partial}{\partial h}, \frac{\partial}{\partial t} \right) (\dot{X}'), X' \right\rangle'_{(h,t)} dt.$$

To integrate by parts a second time, note that

$$\frac{\partial}{\partial t} \langle W', \nabla'_{\partial/\partial t} X' \rangle' = \langle \nabla'_{\partial/\partial t} W', \nabla'_{\partial/\partial t} X' \rangle' + \langle W', \nabla'^2_{\partial/\partial t} X' \rangle',$$

and that  $\frac{1}{2} \partial \Phi(v'(h)) / \partial h$  therefore becomes

$$-\left[ \langle W', \nabla'_{\partial/\partial t} X' \rangle'_{(h,t)} \right]_0^1 + \int_0^1 \langle W', \nabla'^2_{\partial/\partial t} X' \rangle'_{(h,t)} dt \\ + \int_0^1 \left\langle R' \left( \frac{\partial}{\partial h}, \frac{\partial}{\partial t} \right) (\dot{x}), X' \right\rangle'_{(h,t)} dt.$$

But the last term can be written as

$$\int_0^1 \langle R(W, \dot{x})(\dot{x}), X \rangle_{v(h,t)} dt,$$

because  $\nabla'$  is induced from  $\nabla$  by  $v$ .

Now  $v'$  is a variation through curves whose initial and final points are fixed, so that  $W'_{t=0}$  and  $W'_{t=1}$  are both 0. So the first term vanishes again, and

$$\frac{1}{2} \frac{\partial \Phi(v'(h))}{\partial h} = \int_0^1 \langle W', \nabla'^2_{\partial/\partial t} X' \rangle'_{(h,t)} dt + \int_0^1 \langle R(W, \dot{x})(\dot{x}), X \rangle_{v(h,t)} dt.$$

Because  $R$  is the Riemann tensor of the Levi-Civita connection defined by  $\langle \bullet, \bullet \rangle$ , it follows that

$$\langle R(W, \dot{x})(\dot{x}), X \rangle = \langle R(X, \dot{x})(\dot{x}), W \rangle,$$

by Hawking & Ellis (1973: eqn (2.27b)) for example. So

$$\left[ \frac{1}{2} \frac{\partial \Phi(v(h))}{\partial h} \right]_{h=0} = \int_0^1 \langle W, \nabla^3_{\partial/\partial t} \dot{x} + R(\nabla_{\partial/\partial t} \dot{x}, \dot{x})(\dot{x}) \rangle_{x(t)} dt.$$

The proof is completed by contradiction, in a standard way. If, for some  $t \in (0, 1)$ , we have  $\nabla^3_{\partial/\partial t} \dot{x} + R(\nabla_{\partial/\partial t} \dot{x}, \dot{x})(\dot{x}) \neq 0$ , we must describe a variation  $v'$  such that

$$\int_0^1 \langle W, \nabla^3_{\partial/\partial t} \dot{x} + R(\nabla_{\partial/\partial t} \dot{x}, \dot{x})(\dot{x}) \rangle_{x(t)} dt \neq 0.$$

But a variation may be chosen to yield any  $W$  we like, at least along some small open interval containing  $t$ .

#### 4. A special case

We now specialize the Theorem 1 to the case where  $M$  is the Lie group  $G = SO(3)$ . The Lie algebra  $\mathfrak{so}(3)$  of  $G$  is the space of  $3 \times 3$  skew-symmetric matrices and is generated by

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

and  $[E_1, E_2] = E_3$ ,  $[E_2, E_3] = E_1$ , and  $[E_3, E_1] = E_2$ .

We form an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{so}(3)$  by declaring  $\{E_1, E_2, E_3\}$  to be an orthonormal basis, and extend this by right multiplication to a right-invariant Riemannian metric  $\langle \cdot, \cdot \rangle_R$  defined over the whole of  $G$ ; the compatible symmetric connection  $\nabla$  on  $TG$  is then right-invariant. (Note that  $\langle \cdot, \cdot \rangle_R$  is also left-invariant, because  $\langle \cdot, \cdot \rangle_R$  is invariant with respect to the adjoint action of  $\mathfrak{so}(3)$  on itself.)

For  $g \in G$ , let  $R(g) : G \rightarrow G$  be right multiplication. Let  $\bar{E}_i$  be the smooth vector field on  $G$  given by  $\bar{E}_i(g) = dR(g)_e(E_i)$ , where  $i = 1, 2, 3$  and  $e$  is the identity. Now  $\{\bar{E}_1(g), \bar{E}_2(g), \bar{E}_3(g)\}$  is an orthonormal basis of  $(TG)_g$ , and so, by Hawking & Ellis (1973: p. 48, formula (2)),

$$2\langle \nabla_{\bar{E}_i} \bar{E}_i, \bar{E}_k \rangle_R = \langle \bar{E}_j, [\bar{E}_k, \bar{E}_i] \rangle_R + \langle \bar{E}_i, [\bar{E}_k, \bar{E}_j] \rangle_R - \langle \bar{E}_k, [\bar{E}_i, \bar{E}_j] \rangle_R.$$

But  $\langle \cdot, \cdot \rangle_R$ ,  $\bar{E}_i$ ,  $\bar{E}_j$ , and  $\bar{E}_k$  are right-invariant, and the Lie bracket operation is compatible with the diffeomorphism  $R(g)$ . Therefore

$$\begin{aligned} 2\langle \nabla_{\bar{E}_i} \bar{E}_i, \bar{E}_k \rangle_R &= \langle E_j, [E_k, E_i] \rangle + \langle E_i, [E_k, E_j] \rangle - \langle E_k, [E_i, E_j] \rangle \\ &= -\langle E_k, [E_i, E_j] \rangle, \end{aligned}$$

since  $\langle \cdot, \cdot \rangle$  is invariant with respect to the adjoint action of  $\mathfrak{so}(3)$ . Therefore  $(\nabla_{\bar{E}_i} \bar{E}_i)_g = -\frac{1}{2}(dR(g))_e [E_i, E_j] = -\frac{1}{2}[\bar{E}_i, \bar{E}_j]_g$ .

LEMMA 1.  $R(\bar{E}_i, \bar{E}_j)(\bar{E}_k) = -\frac{1}{4}[[\bar{E}_i, \bar{E}_j], \bar{E}_k]$ .

*Proof.*

$$\begin{aligned} R(\bar{E}_i, \bar{E}_j)(\bar{E}_k) &= \nabla_{\bar{E}_i} \nabla_{\bar{E}_j} \bar{E}_k - \nabla_{\bar{E}_j} \nabla_{\bar{E}_i} \bar{E}_k - \nabla_{[\bar{E}_i, \bar{E}_j]} \bar{E}_k \\ &= -\frac{1}{2} \nabla_{\bar{E}_i} [\bar{E}_k, \bar{E}_j] + \frac{1}{2} \nabla_{\bar{E}_j} [\bar{E}_k, \bar{E}_i] + \frac{1}{2} [E_k, [\bar{E}_i, \bar{E}_j]], \end{aligned}$$

since  $[\bar{E}_i, \bar{E}_j]$  is of the form  $\bar{E}_q$  for some  $q$ . So we obtain

$$\begin{aligned} \frac{1}{4} [[\bar{E}_k, \bar{E}_j], \bar{E}_i] - \frac{1}{4} [[\bar{E}_k, \bar{E}_i], \bar{E}_j] + \frac{1}{2} [E_k, [\bar{E}_i, \bar{E}_j]] &= \\ \frac{1}{4} ([\bar{E}_k, \bar{E}_j], \bar{E}_i) + ([\bar{E}_i, \bar{E}_k], \bar{E}_j) + ([\bar{E}_j, \bar{E}_i], \bar{E}_k) + \frac{1}{4} [\bar{E}_k, [\bar{E}_i, \bar{E}_j]]. \end{aligned}$$

According to the Jacobi identity, we are left with the last term.

Let  $x : C \rightarrow G$  be a smooth map, where  $C$  is a smooth manifold possibly with boundary. Then any smooth lifting  $V$  of  $x$  to  $TG$  can be written in the form

$$V = v_i \bar{E}_i \circ x.$$

Here summation is understood, and the  $v_i$  are smooth real-valued functions on  $C$ . Take  $C$  to be  $[0, 1]$  and let  $V(t) = dx_t(d/dt) = \dot{x}(t)$ .

LEMMA 2.

- (1)  $\nabla_{d/dt} V = \frac{dv_i}{dt} \bar{E}_i \circ x$ ;
- (2)  $\nabla_{d/dt}^2 V = \frac{d^2 v_i}{dt^2} \bar{E}_i \circ x - \frac{1}{2} \frac{dv_i}{dt} v_j [\bar{E}_i, \bar{E}_j]_x$ ;
- (3)  $\nabla_{d/dt}^3 V = \frac{d^3 v_i}{dt^3} \bar{E}_i \circ x - \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x + \frac{1}{4} \frac{dv_i}{dt} v_j v_k [[\bar{E}_i, \bar{E}_j], \bar{E}_k]_x$ .

*Proof.*

$$(1) \quad \nabla_{d/dt} V = \frac{dv_i}{dt} \bar{E}_i \circ x + v_i \nabla_{d/dt} (\bar{E}_i \circ x) = \frac{dv_i}{dt} \bar{E}_i \circ x + v_i v_j (\nabla_{\bar{E}_i} \bar{E}_j)_x \\ = \frac{dv_i}{dt} \bar{E}_i \circ x - \frac{1}{2} v_i v_j [\bar{E}_i, \bar{E}_j]_x.$$

But the second terms sum to zero because the Lie bracket is skew-symmetric.

$$(2) \quad \nabla_{d/dt}^2 V = \frac{d^2 v_i}{dt^2} \bar{E}_i \circ x + \frac{dv_i}{dt} \nabla_{d/dt} (\bar{E}_i \circ x) = \frac{d^2 v_i}{dt^2} \bar{E}_i \circ x + \frac{dv_i}{dt} v_j (\nabla_{\bar{E}_i} \bar{E}_j)_x \\ = \frac{d^2 v_i}{dt^2} \bar{E}_i \circ x - \frac{1}{2} \frac{dv_i}{dt} v_j [\bar{E}_i, \bar{E}_j]_x.$$

$$(3) \quad \nabla_{d/dt}^3 V = \frac{d^3 v_i}{dt^3} \bar{E}_i \circ x + \frac{d^2 v_i}{dt^2} \nabla_{d/dt} (\bar{E}_i \circ x) - \frac{1}{2} \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x \\ - \frac{1}{2} \frac{dv_i}{dt} \frac{dv_j}{dt} [\bar{E}_i, \bar{E}_j]_x - \frac{1}{2} \frac{dv_i}{dt} v_j \nabla_{d/dt} [\bar{E}_i, \bar{E}_j]_x \\ = \frac{d^3 v_i}{dt^3} \bar{E}_i \circ x + \frac{d^2 v_i}{dt^2} v_j (\nabla_{\bar{E}_i} \bar{E}_j)_x - \frac{1}{2} \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x \\ - \frac{1}{2} \frac{dv_i}{dt} \frac{dv_j}{dt} [\bar{E}_i, \bar{E}_j]_x - \frac{1}{2} \frac{dv_i}{dt} v_j v_k \nabla_{\bar{E}_k} [\bar{E}_i, \bar{E}_j]_x \\ = \frac{d^3 v_i}{dt^3} \bar{E}_i \circ x - \frac{1}{2} \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x - \frac{1}{2} \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x \\ - \frac{1}{2} \frac{dv_i}{dt} \frac{dv_j}{dt} [\bar{E}_i, \bar{E}_j]_x + \frac{1}{4} \frac{dv_i}{dt} v_j v_k [[\bar{E}_i, \bar{E}_j], \bar{E}_k]_x \\ = \frac{d^3 v_i}{dt^3} \bar{E}_i \circ x - \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x - \frac{1}{2} \frac{dv_i}{dt} \frac{dv_j}{dt} [\bar{E}_i, \bar{E}_j]_x \\ + \frac{1}{4} \frac{dv_i}{dt} v_j v_k [[\bar{E}_i, \bar{E}_j], \bar{E}_k]_x,$$

where the third term sums to zero, again by skew-symmetry of the Lie bracket.

Consider the variational problem described in Section 3. In the present situation there is little to be lost by choosing  $x_0$  to be the identity of  $G$ , and then  $v_0 \in \mathfrak{so}(3)$ . Indeed  $v_0 = v_i(0)E_i$ . Because induction is functorial, Theorem 1 says that  $x$  is a critical point of  $\Phi$  precisely when

$$\nabla_{d/dt}^3 V + R(\nabla_{d/dt} V, V)(V) = 0. \quad (4.1)$$

Since  $R$  is a tensor, and according to Lemmas 2 and 1 above, (4.1) says that

$$\frac{d^3 v_i}{dt^3} \bar{E}_i \circ x - \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_x + \frac{1}{4} \frac{dv_i}{dt} v_j v_k [[\bar{E}_i, \bar{E}_j], \bar{E}_k]_x \\ - \frac{1}{4} \frac{dv_i}{dt} v_j v_k [[\bar{E}_i, \bar{E}_j], \bar{E}_k]_x = 0,$$

namely

$$\frac{d^3 v_i}{dt^3} \bar{E}_i \circ \mathbf{x} - \frac{d^2 v_i}{dt^2} v_j [\bar{E}_i, \bar{E}_j]_{\mathbf{x}} = 0,$$

or rather

$$\begin{aligned} \frac{d^3 v_1}{dt^3} &= \frac{d^2 v_2}{dt^2} v_3 - \frac{d^2 v_3}{dt^2} v_2, & \frac{d^3 v_2}{dt^3} &= \frac{d^2 v_3}{dt^2} v_1 - \frac{d^2 v_1}{dt^2} v_3, \\ \frac{d^3 v_3}{dt^3} &= \frac{d^2 v_1}{dt^2} v_2 - \frac{d^2 v_2}{dt^2} v_1. \end{aligned}$$

We write  $\mathbf{v} = (v_1, v_2, v_3)$ . Then  $\mathbf{v}$  is a curve in  $\mathbb{R}^3$  satisfying

$$\frac{d^3 \mathbf{v}}{dt^3} = \frac{d^2 \mathbf{v}}{dt^2} \times \mathbf{v}$$

where  $\times$  denotes vector product in  $\mathbb{R}^3$ . Equivalently, since

$$\frac{d}{dt} \left( \frac{d\mathbf{v}}{dt} \times \mathbf{v} \right) = \frac{d^2 \mathbf{v}}{dt^2} \times \mathbf{v} + \frac{d\mathbf{v}}{dt} \times \frac{d\mathbf{v}}{dt} = \frac{d^2 \mathbf{v}}{dt^2} \times \mathbf{v},$$

we may write

$$\frac{d^2 \mathbf{v}}{dt^2} = \frac{d\mathbf{v}}{dt} \times \mathbf{v} + \mathbf{c}, \quad (4.2)$$

where  $\mathbf{c} \in \mathbb{R}^3$  is an arbitrary constant vector. It is unlikely that the general solution of (4.2) can be given in closed form, but it is not hard to obtain some families of solutions, including periodic orbits.

## 5. Conclusion

In this paper, we present a natural method for interpolating points by curves subject to nonlinear constraints. When  $M = \mathbb{R}$ , our functional  $\Phi$  is minimized by the familiar cubic splines, and consequently our interpolating curves are generalizations of cubic splines. Splining on  $SO(3)$  arises in a natural way in robotics, and  $\Phi$  is chosen with an engineering application in mind.

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