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ON THE EXPONENTIAL CONVERGENCE RATE OF NONLINEAR ROBOT CONTROLLERS USING THE "PR MODIFICATION"

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Abstract

In this article we analyze a class of positive-real tracking controllers for rigid-link manipulators. In particular, we show that the positive-real controller can be strictly proper and we develop bounds on the rate of convergence of our scheme. This new stability result is important in the design of control laws as it allows the control system designer to roll off the controller gains at high frequencies so that unmodeled dynamics can be justifiably ignored.

Key Words

Robot contol, Lyapunov stability

1. Introduction

In this article we show that a robot controller consisting of a strictly proper positive-real (PR) controller together with feedforward of the robot dynamics for the desired trajectory is exponentially stable. This is an extension of our previous work, in which we required a certain amount of damping at high frequencies in the controller [1]. We also derive bounds on the convergence rate for this new controller and show explicitly how this rate depends on the low-frequency behavior of the controller normalized by the inertia of the manipulator. These results give the control system designer a better understanding of how controller parameters affect the dynamic performance of the robot/controller system. Also, this new controller has the benefits of any passive controller. Specifically, it does not cause limit cycling in the presence of friction.

The analysis of our PR modification and earlier stability proofs of the PD *plus* dynamic compenation control schemes rely heavily on finding Lyapunov functions derived from the mechanical energy of the manipulator. The simplest such scheme is a set-point control scheme developed by Takegaki and Arimoto [2]. The typical difficulty in the stability analysis of these schemes is the lack of negative definiteness in the derivative of the Lyapunov functions. In the set-point control case Takegaki and Arimoto were able to circumvent this problem by applying LaSalle's Theorem. However, in the tracking control case, the error dynamics are no longer autonomous and LaSalle's theorem is not applicable. This technical problem was overcome by Paden and Panja [3] by applying a generalization of LaSalle's theorem proved by Matrosov [4]. A drawback of this technique is that it has not proved effective on the adaptive case.

An alternative approach that leads to a good adaptive control scheme was proposed by Slotine and Li [5]. This scheme eliminated the non-negative definiteness problem by incorporating variable structure control ideas. More recently, exponentially stable adaptive controllers have been developed by Bayard and Wen [6,7] and Sadegh and Horowitz [8]. Exponential stability in the tracking error was achieved by the introduction of certain cross terms in the position and velocity error to the Lyapunov function. This technique was first applied to the robot control problem by Arimoto and Miyazaki to prove exponential stability of set-point controllers [9].

These recent results have improved our global understanding of the robot/controller dynamics immensely. However, when looking at the linearization of these control laws, we recognize that we just have PD control. In this article we show that we can overcome the constraints imposed by the "PD+" control schemes by making an extension from PD joint controllers to positive-real joint controllers. We call this extension the *PR modification.*

The importance of passive joint control laws to which we add dynamic compensation should be emphasized. Any passive control law, including PD control, has the property of not generating limit cycles in the presence of joint friction and stiction. This is well known for PD controllers, but it is precisely the fact that PD controllers are *passive* that makes them unable to dump power continuously into friction losses in a limit cycle. PD controllers are only a small subset of the set of PR controllers that we define, all of which do not limit cycle in the presence of friction or stiction. Finally, this PR modification can be applied to the adaptive versions of the "PD+" control laws as well [6,8].

The format of this article is as follows. In Section 2 we introduce our notation. This notation makes obvious the symmetry properties of the Coriolis forces that are inherited from the inertia tensor. In Section 3 we introduce a class of positive-real controllers that are important in servo control, and in Section 4 these controllers are incorporated into a nonlinear manipulator controller with dynamic compensation. Our conclusions are given in Section 5.

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2. Manipulator Dynamics

In this section we derive the dynamics of an m-joint rigid-link robot manipulator using tensor notation. This general construction is important as it preserves the structure of the dynamics very well; the fictitious forces are described explicitly in terms of the intertia tensor, and their multilinearity is clear. We use the following notation.

- R real numbers
- R_+ non-negative real numbers
- $[M]$ the matrix whose components are the same as the covariant tensor M of order 2 (in other words, $\alpha^T\beta \triangleq M(\alpha,\beta)$)
- $\langle x,y\rangle$ usual scalar product of $x,y \in R^n$
- D_x derivative with respect to x
- Z^* conjugate transpose of the matrix Z
- $F(\cdot)$ the map $F(\cdot) : \alpha \rightarrow F(\alpha)$

Let $\theta \in R^m$ be the vector of manipulator joint displacements and denote the configuration dependent inertia tensor of the robot manipulator by $M(\theta)$. We make *assumption 1* that $M(\theta)$ is symmetric, positive definite, and twice continuously differentiable in some open region of its configuration space. Denote the components of $M(\theta)$ by $M(\theta)_{ij}$. That is,

$$M(\theta)(\alpha,\beta) = M(\theta)_{ij}\alpha^i\beta^i \tag{1}$$

where α^i are the components of $\alpha \in R^n$, and the summation over i and j in the right-hand side is implied. The derivative of $M(\theta)$ with respect to θ is a covariant tensor of order 3 with components

$$D_\theta M(\theta)_{ijk} = \frac{\partial M(\theta)_{ij}}{\partial\theta^k} \tag{2}$$

We define

$$D_\theta M(\theta)(\alpha,\beta)(\gamma) \triangleq D_\theta M(\theta)_{ijk}\alpha^i\beta^j\gamma^k \tag{3}$$

and

$$D_\theta^2 M(\theta)(\alpha,\beta)(\gamma)(\delta) \triangleq \frac{\partial^2 M(\theta)_{ij}}{\partial\theta^k\partial\theta^l}\alpha^i\beta^j\gamma^k\delta^l \tag{4}$$

Thus, $D_\theta M(\theta)$ is linear in each of the three arguments α, β, and γ, as is $D_\theta^2 M(\theta)$ in its four. In addition, let $G(\theta)$ denote the potential energy due to gravity for the configuration θ. We make *assumption 2* that $G(\theta)$ is continuously differentiable and that this derivative is bounded. The Lagrangian for a manipulator is therefore

$$L = \frac{1}{2}M(\theta)(\dot\theta,\dot\theta) - G(\theta) \tag{5}$$

The joint forces are given by

$$F = \frac{d}{dt}D_{\dot\theta}L - D_\theta L \tag{6}$$

In this formalism F is a linear operator (covariant tensor of order 1) that *acts* on a displacement to do work. From (5) and (6) we have

$$F(\cdot) = \frac{d}{dt}M(\theta)(\dot\theta,\cdot) - \frac{1}{2}D_\theta M(\theta)(\dot\theta,\dot\theta)(\cdot) + D_\theta G(\theta)(\cdot) \tag{7}$$

$$= M(\theta)(\ddot\theta,\cdot) + D_\theta M(\theta)(\dot\theta,\cdot)(\dot\theta) - \frac{1}{2}D_\theta M(\theta)(\dot\theta,\dot\theta)(\cdot) + D_\theta G(\theta)(\cdot) \tag{8}$$

The Coriolis and centrifugal forces, those quadratic in $\dot\theta$, are sometimes expressed as $< C(\theta,\dot\theta)\dot\theta,\cdot >$ where $C(\theta,\dot\theta)$ is a matrix defined by

$$\alpha^T C(\theta,\dot\theta)\beta = \frac{1}{2}D_\theta M(\theta)(\dot\theta,\alpha)(\beta) + \frac{1}{2}D_\theta M(\theta)(\beta,\alpha)(\dot\theta) - \frac{1}{2}D_\theta M(\theta)(\dot\theta,\beta)(\alpha) \quad \forall\alpha,\beta\in R^m \tag{9}$$

It is easy to verify that

$$\frac{d}{dt}[M(\theta)] - 2C(\theta,\dot\theta) \tag{10}$$

is skew-symmetric. In this article, the dynamics will be used in the form (8).

3. Mathematical Preliminaries

In this section we describe the class of "*PR* controllers" th^t are useful in the robot control problem. They are also important for the servo problem in general, as these passive controllers are immune to limit cycling in the presence of friction and stiction. In addition to the passive PR property, we require that these controllers produce a restoring force similar to proportional feedback. We give a precise definition of these controllers after the following definition and lemmas.

Definition. A square rational transfer function matrix $Z(s)$ is said to be *positive real* if (a) $Z($ $+ Z^*(s)$ is non-negative definite Hermitian for Re $s > 0$; (b) the elements of $Z(s)$ have no poles in the open right half-plane, and the $j\omega$-axis poles are simple and such that the associated residue matrix is non-negative definite.

PR Lemma [10]. Let $Z(\cdot)$ be a matrix of rational transfer functions such that $Z(\infty)$ is finite and Z has poles that lie in Re $s < 0$ or are simple on Re $s = 0$. Let $[A,B,C,Z(\infty)]$ be a minimal realization of Z. Then $Z(\cdot)$ is positive real if and only if there exists a symmetric positive definite P and matrices W_0 and L such that

$$PA + A^TP = -LL^T \tag{11}$$

$$PB = C^T - LW_0 \tag{12}$$

$$W_0^TW_0 = Z(\infty) + Z^T(\infty) \tag{13}$$

It is useful to pin down the dimensions of the matrices in the realization $[A,B,C,Z(\infty)]$. Let $A \in R^{n\times n}$; $B \in R^{n\times m}$, where m is also the number of degrees of freedom of the manipulator; $C \in R^{m\times n}$; and $Z(\infty) \in R^{m\times m}$. Also, note that by a change of coordinates by $\sqrt{P}$ we can replace P with the identity matrix. We will do this without any loss of generality.

Lemma 1. Let $Z(s)$ be positive real. Then $Z(s)$ can be uniquely decomposed into lossless and non-lossless components

$$Z(s) = Z_N(s) + Z_L(s) \tag{14}$$

where $Z_L(s)$, the "lossless component of $Z(s)$", is the sum of the terms in the partial fraction of $Z(s)$ corresponding to poles on the $j\omega$-axis (including a possible pole at infinity). Both $Z_N(s)$ and $Z_L(s)$ are positive real.

Proof. See [11], Section 5.2.

In this article a proper subset of all positive real transfer functions is important and is defined in the terms of Lemma 1.

Definition 2. A square matrix $Z(\cdot)$ of proper rational transfer functions is called a *PR controller* if (a) $Z(\cdot)$ is positive-real, (b) the non-lossless part of $Z(s)$ is strictly positive real [12], and (c) the only $j\omega$-axis poles of $Z(\cdot)$ are at zero with the associated residue matrix positive definite.

Example. PD controllers when viewed as transfer functions from velocity to force are PR controllers. Let

$$Z(s) \triangleq \frac{K_p}{s} + K_d \tag{15}$$

where the proportional gain K_p and the derivative gain K_d are positive definite when symmetrized. Note that the residue at $s = 0$ is K_p. The remaining conditions are easily verified.

From Lemma 1 we see that any PR controller $Z(s)$ can be written as

$$Z(s) = \frac{K_p}{s} + Z_N(s) \tag{16}$$

where K_p is the residue at zero having the interpretation of low-frequency proportional gain, and $Z_N(\infty) = Z(\infty)$.

A PR controller is passive in a real mechanical energy sense when its input is a velocity and its output is a force directed opposite the velocity vector. We will use PR controllers in this way.

4. Proving Stability with PR Controllers

In this section we introduce a nonlinear control law for robot manipulators that is based on the strictly proper PR controller. This controller consists of a nonlinear feedforward of the nominal dynamics computed (possibly off-line) along a desired joint trajectory θ_d, plus linear feedback of the tracking error through a PR controller $Z(s)$. We make *assumption 3* that the desired trajectory is twice continuously differentiable with both of these derivatives bounded.

The control law is given by

$$F(\cdot) = M(\theta_d)(\ddot{\theta}_d, \cdot) + D_\theta M(\theta_d)(\dot{\theta}_d, \cdot)(\dot{\theta}_d)$$

$$-\frac{1}{2} D_\theta M(\theta_d)(\dot{\theta}_d, \dot{\theta}_d)(\cdot) + D_\theta G(\theta)(\cdot) - \langle Cz, \cdot\rangle - \langle K_p\tilde{\theta}, \cdot\rangle \tag{17}$$

where z is the state of the controller; K_p is the residue of $Z(s)$ at zero; and $[A, B, C]$ is the realization of $Z_N(s)$, the non-lossless part of $Z(s)$ given by the PR lemma. Before considering the performance of this controller on the nonlinear system with dynamics described by (8), we investigate the behavior of a linear version of this system with the linear feedback described in (17). We do this for several reasons. First, the key part of the result can be expressed without awkward restrictions. Second, the linear result facilitates the statement of the nonlinear result. Finally, the nonlinear result is proved by showing that the linear result is strong enough to bound the extra terms introduced by the nonlinear dynamics. Therefore, presenting the linear result separately reveals the most important aspects of the control system within the simplest framework.

The analysis of the performance of the nonlinear system (8), (17) will be done by introducing the tracking error $(\tilde{\theta}, \dot{\tilde{\theta}}) \triangleq (\theta - \theta_d, \dot{\theta} - \dot{\theta}_d)$ and proving that it converges exponentially to zero using Lyapunov techniques. We capture the heart of this analysis when we consider the linear system

$$M(\ddot{\tilde{\theta}}, \cdot) = F(\cdot) \tag{18}$$

where we choose the feedback part of (17) for F

$$F(\cdot) = -\langle K_p\tilde{\theta}, \cdot\rangle - \langle Cz, \cdot\rangle \tag{19}$$

where z is the state of $Z_N(s)$

$$\dot{z} = Az + B\dot{\tilde{\theta}} \tag{20}$$

The system (18)–(20) is linear because we assume that M in (18) is constant. The exponential stability of the zero solution of (18)–20) is established by the following theorem.

Theorem 1. If $Z_N(s) = C(sI - A)^{-1}B$ is SPR and if $K_p = K_p^T$ is positive definite, then the zero solution of the system (18)–(20) is exponentially stable.

Proof. Nowhere in this proof do we make use of the linearity of (18). Therefore, the proof technique carries over to the nonlinear case. We define

$$K_d \triangleq Z_N(0) = -CA^{-1}B \tag{21}$$

and give the proof under the additional assumption that K_d is symmetric. The proof for the case with nonsymmetric K_d is left as an exercise for the interested reader. (There is little reason to choose nonsymmetric feedback for an inherently symmetric system.) As discussed in the previous section, we can, without loss of generality, take $C = B^T$ and assume that

$$A + A^T = -2Q \text{ where } Q = Q^T > 0 \tag{22}$$

Next we define a natural energy function for the system (18)–(20),

$$V_o = \left(\frac{1}{2}\right)\left(\langle z, z\rangle + \langle K_p\tilde{\theta}, \tilde{\theta}\rangle + M(\dot{\tilde{\theta}}, \dot{\tilde{\theta}})\right) \tag{23}$$

The derivative of V_o along trajectories of (18)–(20) is negative semi-definite,

$$\dot{V}_o^{(18)-(20)} = -\langle z, Qz\rangle \tag{24}$$

This implies that (18)–(20) is stable (in the sense of Lyapunov). This can be strengthened to uniform asymptotic stability using the invariance principle; since the system is linear this implies exponential stability. This approach does not transfer well to the nonlinear case. Therefore we modify this Lyapunov function to obtain one that has a negative definite derivative. We use the Lyapunov function

$$V = V_o - (\frac{1}{2})b\langle z, z\rangle + b\langle A^{-1}z, B\dot{\tilde{\theta}}\rangle + c\langle A^{-1}z, B\tilde{\theta}\rangle \quad (25)$$

$$+(\frac{1}{2})c\langle A^{-T}B\tilde{\theta}, QA^{-T}B\tilde{\theta}\rangle + cM(\dot{\tilde{\theta}}, \tilde{\theta})$$

which can be written as

$$V = (1-a)V_o + \frac{1}{2}\begin{bmatrix} z \\ \tilde{\theta} \\ \dot{\tilde{\theta}} \end{bmatrix}^T \begin{bmatrix} (a-b)I & cA^{-T}B & bA^{-T}B \\ cB^TA^{-1} & aK_p + cK_d & c[M] \\ bB^TA^{-1} & c[M] & a[M] \end{bmatrix} \begin{bmatrix} z \\ \tilde{\theta} \\ \dot{\tilde{\theta}} \end{bmatrix} \quad (26)$$

$$= (1-a)V_o + \frac{1}{2}\begin{bmatrix} z \\ [M]^{1/2}\tilde{\theta} \\ [M]^{1/2}\dot{\tilde{\theta}} \end{bmatrix}^T \begin{bmatrix} (a-b)I & cA^{-T}B[M]^{-1/2} & bA^{-T}B[M]^{-1/2} \\ c[M]^{-T/2}B^TA^{-1} & [M]^{-T/2}(aK_p + cK_d)[M]^{-1/2} & CI \\ b[M]^{-T/2}B^TA^{-1} & cI & aI \end{bmatrix} \begin{bmatrix} z \\ [M]^{1/2}\tilde{\theta} \\ [M]^{1/2}\dot{\tilde{\theta}} \end{bmatrix}$$

is positive definite for any $a \in (0,1)$ if the matrix in (26) is positive semidefinite. The derivative of V along the trajectories of (18)–(20) is given by

$$\dot{V}^{(18)-(20)} = \langle z, AZ\rangle + \langle z, B\dot{\tilde{\theta}}\rangle$$

$$+\langle K_p\tilde{\theta}, \dot{\tilde{\theta}}\rangle - \langle K_p\tilde{\theta}, \dot{\tilde{\theta}}\rangle - \langle Cz, \dot{\tilde{\theta}}\rangle$$

$$-b\langle z, Az\rangle - b\langle z, B\dot{\tilde{\theta}}\rangle$$

$$+b\langle z, B\dot{\tilde{\theta}}\rangle + b\langle A^{-1}B\dot{\tilde{\theta}}, B\dot{\tilde{\theta}}\rangle \quad (27)$$

$$-b\langle A^{-1}z, B[M]^{-1}K_p\tilde{\theta}\rangle - b\langle A^{-1}z, B[M]^{-1}Cz\rangle$$

$$+c\langle z, B\tilde{\theta}\rangle + c\langle A^{-1}B\dot{\tilde{\theta}}, B\tilde{\theta}\rangle + c\langle A^{-1}z, B\dot{\tilde{\theta}}\rangle$$

$$+c\langle A^{-T}B\tilde{\theta}, QA^{-T}B\dot{\tilde{\theta}}\rangle$$

$$+cM(\dot{\tilde{\theta}}, \dot{\tilde{\theta}}) - c\langle K_p\tilde{\theta}, \tilde{\theta}\rangle - c\langle Cz, \tilde{\theta}\rangle$$

Making use of the fact that $C = B^T$ and the symmetry of $B^TA^{-1}B$, this expression can be simplified:

$$\dot{V} = \langle z, (A - bA - bA^{-T}B[M]^{-1}C)z\rangle +$$

$$\langle z, -bA^{-T}B[M]^{-1}K_p\tilde{\theta}\rangle \quad (28)$$

$$+\langle z, -cA^{-T}B\dot{\tilde{\theta}}\rangle + \langle -cK_p\tilde{\theta}, \tilde{\theta}\rangle + \langle\dot{\tilde{\theta}}, (-bK_d + c[M])\dot{\tilde{\theta}}\rangle$$

The fact that we can choose b and c so that V is positive definite and $\dot{V}^{(18)-(20)}$ is negative definite is not immediately obvious. Therefore, we complete the proof by finding specific values of b and c that give the desired sign definiteness. We define several system-dependent constants that will be used to determine appropriate values of b and c.

$$\underline{\beta_p} = \underline{\sigma}([M]^{-T/2}K_p[M]^{-1/2})$$
$$\beta_p = \overline{\sigma}([M]^{-T/2}K_p[M]^{-1/2})$$
$$\underline{\beta_d} = \underline{\sigma}([M]^{-T/2}K_d[M]^{-1/2})$$
$$\beta_d = \overline{\sigma}([M]^{-T/2}K_d[M]^{-1/2})$$
$$\gamma = 3/\beta_d$$
$$\beta_1 = \overline{\sigma}(A^{-T}B[M]^{-1/2})$$
$$\beta_2 = \gamma\underline{\sigma}(Q) + \gamma\beta_1^2\overline{\sigma}(A) + \frac{1}{2}$$

(29)

where $\overline{\sigma}$ and $\underline{\sigma}$ denote the maximum and minimum singular value of a matrix respectively.

Remark. The conservatism of our estimate of the exponential decay rate is reduced when upper bound $\overline{\beta}_p$ and lower bound $\underline{\beta_p}$ are close to the same value and when $\overline{\beta}_d$ and $\underline{\beta_d}$ are close to the same value. This requires that K_p and K_d should have the same "shape" as $[M]$. It is also valid to define these constants using $[M]^{-1}$ on one side of K_p and K_d rather than using the square root on both sides as above. This leads naturally to the computed torque scheme in the nonlinear case.

We simplify our construction to a single variable by taking $b = \gamma c$, which allows us to rewrite (28) as

$$\dot{V}^{(18)-(20)} = -\langle z, (Q - c\gamma Q + c\gamma A^{-T}B[M]^{-1}B^T)z\rangle -$$

$$c\langle z, \gamma A^{-T}B[M]^{-1}K_p\tilde{\theta}\rangle \quad (30)$$

$$-c\langle z, A^{-T}B\dot{\tilde{\theta}}\rangle - c\langle K_p\tilde{\theta}, \tilde{\theta}\rangle - c\langle\dot{\tilde{\theta}}, (\gamma K_d - [M])\dot{\tilde{\theta}}\rangle$$

It is then an exercise to show that V satisfies, for sufficiently small positive c,

from which it is obvious that $\dot{V}^{(18)-(20)}$ is negative definite for all sufficiently small positive c. Since it is also obvious that V is positive for c sufficiently small, this establishes the assertion of the theorem. However, after having worked so hard to obtain expressions (26) and (31), we wish to say more than simply "for c sufficiently small." Furthermore, from (31) we see that c can be used as an estimate of the exponential decay rate; hence, we would like to choose c as large as possible.

Let a be any constant in the interval (0,1) and define c^* as the smallest positive solution of

$$
\dot{V}^{(18)-(20)} \leq -cV - \frac{1}{2}\begin{bmatrix} |z| \\ |[cM]^{1/2}\tilde{\theta}| \\ |[cM]^{1/2}\tilde{\theta}| \\ |[cM]^{1/2}\dot{\tilde{\theta}}| \end{bmatrix}^T \begin{bmatrix} 2\underline{\sigma}(Q) - 2\beta_2 c + \gamma c^2 & -\beta_1(\gamma\overline{\beta}_p - c)c^{1/2} & -\beta_1(1-\gamma c)c^{1/2} \\ -\beta_1(\gamma\overline{\beta}_p - c)c^{1/2} & \underline{\beta_p} - \overline{\beta}_d c & -c \\ -\beta_1(1-\gamma c)c^{1/2} & -c & 3 \end{bmatrix} \begin{bmatrix} |z| \\ |[cM]^{1/2}\tilde{\theta}| \\ |[cM]^{1/2}\dot{\tilde{\theta}}| \end{bmatrix} \tag{31}
$$

$$
0 = \det \begin{bmatrix} a - \gamma c^* & -\beta_1 c^* & -\gamma\beta_1 c^* \\ -\beta_1 c^* & \underline{\beta_p} a + \underline{\beta_d} c^* & -c^* \\ -\gamma\beta_1 c^* & -c^* & a \end{bmatrix}
$$

For all $c \leq c^*$ the matrix in (26) is positive semidefinite; hence, V is positive definite. Define c^{**} as the smallest positive solution of

$$
0 = \det \begin{bmatrix} 2\underline{\sigma}(Q) - 2\beta_2 c^{**} + \gamma(c^{**})^2 & -\beta_1(\gamma\overline{\beta}_p - c^{**})(c^{**})^{1/2} & -\beta_1(1-\gamma c^{**})(c^{**})^{1/2} \\ -\beta_1(\gamma\overline{\beta}_p - c^{**})(c^{**})^{1/2} & \underline{\beta_p} - \overline{\beta}_d c^{**} & -c^{**} \\ -\beta_1(1-\gamma c^{**})(c^{**})^{1/2} & -c^{**} & 3 \end{bmatrix} \tag{33}
$$

For all $c \leq c^{**}$ the matrix in (31) is positive semidefinite; hence, $\dot{V}^{(18)-(20)}$ is negative semidefinite. Therefore, we choose

$$
c = \min(c^*, c^{**}) \tag{34}
$$

After c has been chosen, we can vary a in (26) until the matrix is positive semidefinite. Define $\underline{a}$ as the smallest positive solution of

$$
0 = \det \begin{bmatrix} \underline{a} - \gamma c & -\beta_1 c & -\gamma\beta_1 c \\ -\beta_1 c & \underline{\beta_p}\underline{a} + \underline{\beta_d} c & -c \\ -\gamma\beta_1 c & -c & \underline{a} \end{bmatrix} \tag{35}
$$

This value can then be used to give a lower bound on V as a function of V_o. Obviously $\underline{a} \leq a$. Define $\overline{a}$ to be used in an upper bound on V as the smallest positive solution of

$$
0 = \det \begin{bmatrix} \overline{a} + \gamma c & -\beta_1 c & -\gamma\beta_1 c \\ -\beta_1 c & \underline{\beta_p}\overline{a} - \overline{\beta}_d c & -c \\ -\gamma\beta_1 c & -c & \overline{a} \end{bmatrix} \tag{36}
$$

It follows from (26) and (31)–(36) that along trajectories of (18)–(20), V_o and V satisfy

$$
(1 - \underline{a})V_o \leq V \leq (1 + \overline{a})V_o \tag{37}
$$

$$
\dot{V}^{(18)-(20)} \leq -cV \tag{38}
$$

and

$$
V_o(t) \leq \frac{1+\overline{a}}{1-\underline{a}} V_o(0)\exp[-ct] \tag{39}
$$

Equations (39) and (24) together imply that

$$
V_o(t) \leq \min\left[1, \frac{1+\overline{a}}{1-\underline{a}}\exp[-ct]\right] V_o(0) \tag{40}
$$

This completes the proof and provides an estimate of the rate of decay. QED.

Remarks. The design trade-offs in the choices of A, B, C, and K_p are contained in the equations that define c^* and c^{**}. While these are nontrivial algebraic expressions, they give us an analytical tool for evaluating designs without doing extensive simulations. Equation (37) can be interpreted as a description of the eccentricity and rotation of the V equals a constant ellipsoids with respect to the energy function V_o equals a constant ellipsoids. The constant a in (32) is provided so that we can limit this eccentricity and rotation. When $c^* \leq c^{**}$ a larger rate of exponential decay can be obtained by choosing a larger value for a; that is, the larger estimate of the decay rate is obtained by allowing more eccentricity and rotation. Choosing a larger a leads to larger $\underline{a}$ and larger $\overline{a}$, which implies that the constant multiplying $V_o(0)$ in (39) is also larger. Finally, note that (39) does not preclude an increase in the energy V_o of the system before eventually decaying to zero. The significance of (40) is that it shows that the energy of the system does not increase before decaying to zero.

In the previous theorem we showed that any positive definite feedback on position combined with any SPR feedback on the velocity made the zero solution of (18)–(20) exponentially stable. In this corollary we show that appropriate scaling of the feedback system matrices allows c, our bound on the decay rate, to become arbitrarily large.

Corollary. Let $A', B', C' = (B')^T$ be the state space representation of any strictly proper SPR transfer function. Let K_p' be any positive definite symmetric matrix. Choose $c^{*'}$, $c^{**'}$, and c' as in (32)–(34). If

$$
A = \lambda A' \quad B = \lambda B'
$$

$$
C = \lambda C' \quad K_p = \lambda^2 K_p' \tag{41}
$$

then for all $\lambda \geq 1$, the zero solution of (18)–(20) is exponentially stable and equations (37)–(40) are satisfied with $c = \lambda c'$ and $\overline{a}$ and $\underline{a}$ unchanged.

Proof. Let $\underline{\beta_p'}$ denote the value of $\underline{\beta_p}$ when $\lambda = 1$ and use the corresponding definition of Q', $\overline{\beta}_p'$, $\underline{\beta_d'}$, $\overline{\beta}_d'$, β_1', and β_2'. Then we have

$$
\underline{\beta_p} = \lambda^2 \underline{\beta_p'} \quad \gamma = \lambda^{-1}\gamma'
$$

$$
\beta_p = \lambda^2 \beta_p' \quad \beta_1 = \beta_1'
$$

$$\underline{\beta_d} = \lambda \underline{\beta_d'} \quad \beta_2 = \beta_2'$$
$$\overline{\beta}_d = \lambda \overline{\beta}_d' \quad Q = \lambda Q' \tag{42}$$

The result is established by substituting the expressions (42), $c^* = \lambda c^{*'}$, $c^{**} = \lambda c^{**'}$, and $c = \lambda c'$ into (32)-(36) and showing that $c^{*'}$, $c^{**'}$, c', $\underline{a}$, and $\overline{a}$ are independent of λ. QED.

We are now ready to analyze the performance of the nonlinear system (8), (17), (20). As mentioned above, the analysis is done using the Lyapunov functions V_o and V defined in (23) and (25), respectively. We begin by deriving an equation that describes the motion of the tracking error $\tilde{\theta}$,

$$\begin{aligned} M(\theta)(\ddot{\tilde{\theta}}, \cdot) &= M(\theta)(\ddot{\theta}, \cdot) - M(\theta)(\ddot{\theta}_d, \cdot) \\ &= F(\cdot) - D_\theta G(\theta)(\cdot) - D_\theta M(\theta)(\dot{\theta}, \cdot)(\dot{\theta}) \\ &\quad + \frac{1}{2} D_\theta M(\theta)(\dot{\theta}, \dot{\theta})(\cdot) - M(\theta)(\ddot{\theta}_d, \cdot) \end{aligned} \tag{43}$$

$$\begin{aligned} &= M(\theta_d)(\ddot{\theta}_d, \cdot) - M(\theta)(\ddot{\theta}_d, \cdot) \\ &\quad + D_\theta M(\theta_d)(\dot{\theta}_d, \cdot)(\dot{\theta}_d) - D_\theta M(\theta)(\dot{\theta}, \cdot)(\dot{\theta}) \\ &\quad + \frac{1}{2} D_\theta M(\theta)(\dot{\theta}, \dot{\theta})(\cdot) - \frac{1}{2} D_\theta M(\theta_d)(\dot{\theta}_d, \dot{\theta}_d)(\cdot) \\ &\quad + D_\theta G(\theta_d)(\cdot) - D_\theta G(\theta)(\cdot) \\ &\quad - \langle K_p \tilde{\theta}, \cdot \rangle - \langle Cz, \cdot \rangle \end{aligned} \tag{44}$$

$$\begin{aligned} &= M(\theta_d)(\ddot{\theta}_d, \cdot) - M(\theta)(\ddot{\theta}_d, \cdot) \\ &\quad + D_\theta M(\theta_d)(\dot{\theta}_d, \cdot)(\dot{\theta}_d) - D_\theta M(\theta)(\dot{\theta}_d, \cdot)(\dot{\theta}_d) \\ &\quad + D_\theta M(\theta)(\dot{\theta}_d, \cdot)(\dot{\theta}) - D_\theta M(\theta)(\dot{\theta}, \cdot)(\dot{\theta}) \\ &\quad + D_\theta M(\theta)(\dot{\theta}_d, \cdot)(\dot{\theta}_d) - D_\theta M(\theta)(\dot{\theta}_d, \cdot)(\dot{\theta}) \\ &\quad + D_\theta M(\theta)(\dot{\theta}_d, \dot{\theta})(\cdot) - D_\theta M(\theta)(\dot{\theta}_d, \dot{\theta}_d)(\cdot) \\ &\quad + \frac{1}{2} D_\theta M(\theta)(\dot{\tilde{\theta}}, \dot{\theta})(\cdot) - \frac{1}{2} D_\theta M(\theta)(\dot{\tilde{\theta}}, \dot{\theta}_d)(\cdot) \\ &\quad + \frac{1}{2} D_\theta M(\theta)(\dot{\theta}_d, \dot{\theta}_d)(\cdot) - \frac{1}{2} D_\theta M(\theta_d)(\dot{\theta}_d, \dot{\theta}_d)(\cdot) \\ &\quad + D_\theta G(\theta_d)(\cdot) - D_\theta G(\theta)(\cdot) \\ &\quad - \langle K_p \tilde{\theta}, \cdot \rangle - \langle Cz, \cdot \rangle \end{aligned} \tag{45}$$

$$\begin{aligned} &= \big(M(\theta_d) - M(\theta)\big)(\ddot{\theta}_d, \cdot) + \big(D_\theta M(\theta_d) - D_\theta M(\theta)\big)(\dot{\theta}_d, \cdot)(\dot{\theta}_d) \\ &\quad - D_\theta M(\theta)(\dot{\tilde{\theta}}, \cdot)(\dot{\theta}) - D_\theta M(\theta)(\dot{\theta}_d, \cdot)(\dot{\tilde{\theta}}) + D_\theta M(\theta)(\dot{\theta}_d, \dot{\tilde{\theta}})(\cdot) \\ &\quad + \frac{1}{2} D_\theta M(\theta)(\dot{\tilde{\theta}}, \dot{\tilde{\theta}})(\cdot) + \frac{1}{2}\big(D_\theta M(\theta) - D_\theta M(\theta_d)\big)(\dot{\theta}_d, \dot{\theta}_d)(\cdot) \\ &\quad + \big(D_\theta G(\theta_d) - D_\theta G(\theta)\big)(\cdot) - \langle K_p \tilde{\theta}, \cdot \rangle - \langle Cz, \cdot \rangle \end{aligned} \tag{46}$$

In Theorem 1 the system dynamics were linear; hence, the exponential stability of the origin of (18)-(20) implied global stability. From (46) we see that the dynamics of the tracking error contain quadratic terms. This forces us to limit ourselves to local stability results. Furthermore, M is now a function of θ, which is a function of time; hence the constants defined in (29), such as the bound $\overline{\beta}_p$, will have to be made uniform over time. This is done by making the bounds uniform over all possible θ. The trick is to define the set of all possible θ in such a way that the bounds have appropriate values, i.e., $\overline{\beta}_p < \infty$ and $\underline{\beta_p} > 0$, and at the same time are not too difficult to compute. The simplest way to do this is to assume a compact configuration space. This assumption is unnecessarily restrictive. Eventually, we will prove the exponential stability of the origin of the system (20), (46) and provide a guaranteed region of attraction. It follows that it will be sufficient to make our bounds hold only for θ in a neighborhood around θ_d. Notice that while this neighborhood is time-varying in the θ space, it can be time-invariant in the $\tilde{\theta}$ space. We assume that $\theta_d(t)$ is known and that $M(\theta)$ satisfies assumption 1 for all θ in a specified neighborhood of $\theta_d(t)$ for all t. For the nonlinear case (29) is replaced by

$$\underline{\beta_p} = \inf_t \inf_{|\tilde{\theta}| \le \delta_1} \underline{\sigma}([M(\theta_d + \tilde{\theta})]^{-T/2} K_p [M(\theta_d + \tilde{\theta})]^{-1/2})$$

$$\overline{\beta}_p = \sup_t \sup_{|\tilde{\theta}| \le \delta_1} \overline{\sigma}([M(\theta_d + \tilde{\theta})]^{-T/2} K_p [M(\theta_d + \tilde{\theta})]^{-1/2})$$

$$\underline{\beta_d} = \inf_t \inf_{|\tilde{\theta}| \le \delta_1} \underline{\sigma}([M(\theta_d + \tilde{\theta})]^{-T/2} K_d [M(\theta_d + \tilde{\theta})]^{-1/2})$$

$$\overline{\beta}_d = \sup_t \sup_{|\tilde{\theta}| \le \delta_1} \overline{\sigma}([M(\theta_d + \tilde{\theta})]^{-T/2} K_d [M(\theta_d + \tilde{\theta})]^{-1/2})$$

$$\gamma = 3/\underline{\beta_d} \tag{47}$$

$$\beta_1 = \sup_t \sup_{|\tilde{\theta}| \le \delta_1} \overline{\sigma}(A^{-T} B [M(\theta_d + \tilde{\theta})]^{-1/2})$$

$$\beta_2 = \gamma \underline{\sigma}(Q) + \gamma \beta_1^2 \overline{\sigma}(A) + \frac{1}{2}$$

$$\beta_3 = \sup_t \sup_{|\tilde{\theta}| \le \delta_1} \overline{\sigma}([M(\theta_d + \tilde{\theta})]^{-1/2})$$

where $|\tilde{\theta}| \le \delta_1$ specifies the desired neighborhood around $\theta_d(t)$. The supremum over t in (47) could be replaced by a supremum over θ_d in some compact set of allowed values.

Because the right-hand side of (46) contains terms that were not in (18), the derivative of the Lyapunov function will have additional terms when compared to (27), (28) and (30), (31). Our crude but effective approach is simply to bound these terms, show that the Lyapunov derivative is negative semidefinite in the required neighborhood of the origin of $(z, \tilde{\theta}, \dot{\tilde{\theta}})$ space, and prove that $(z, \tilde{\theta}, \dot{\tilde{\theta}})$ does not leave this neighborhood. Several of the terms in (46) are quadratic in $(\tilde{\theta}, \dot{\tilde{\theta}})$; hence, we will need to restrict the magnitude of $\tilde{\theta}$ and $\dot{\tilde{\theta}}$. To do this, we introduce the set

$$\Delta = \left\{ (\tilde{\theta}, \dot{\tilde{\theta}}) : |\tilde{\theta}| \le \delta_1 \text{ and } |\dot{\tilde{\theta}}| \le \delta_2 \right\} \tag{48}$$

and define the following bounds over time and the set Δ', which is Δ with the origin removed. These bounds involve a supremum over time. The only time-dependence involved is the dependence of θ_d, $\dot{\theta}_d$, and $\ddot{\theta}_d$ on time. As in (47), the supremum over time can be replaced by a supremum over $(\theta_d, \dot{\theta}_d, \ddot{\theta}_d)$ in a compact set of allowed values.

$$\alpha_1 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(M(\theta_d+\tilde{\theta}) - M(\theta_d))(\ddot{\theta}_d, x)|}{|x|\,|\theta|} \right\} \quad (49)$$

$$\alpha_2 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta M(\theta_d+\tilde{\theta}) - D_\theta M(\theta_d))(\dot{\theta}_d, x)(\dot{\theta}_d)|}{|x|\,|\theta|} \right\}$$

$$\alpha_3 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta M(\theta_d+\tilde{\theta})(\dot{\tilde{\theta}}, x)(\dot{\theta}_d+\dot{\tilde{\theta}})|}{|x|\,|\theta|} \right\} \quad (51)$$

$$\alpha_4 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta M(\theta_d+\tilde{\theta})(\dot{\theta}_d, x)(\dot{\theta})|}{|x|\,|\theta|} \right\} \quad (52)$$

$$\alpha_5 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta M(\theta_d+\tilde{\theta})(\dot{\theta}_d, \dot{\tilde{\theta}})(x)|}{|x|\,|\theta|} \right\} \quad (53)$$

$$\alpha_6 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta M(\theta_d+\tilde{\theta})(\dot{\tilde{\theta}}, \dot{\tilde{\theta}})(x)|}{|x|\,|\theta|} \right\} \quad (54)$$

$$\alpha_7 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta M(\theta_d+\tilde{\theta}) - D_\theta M(\theta_d))(\dot{\theta}_d, \dot{\theta}_d)(x)|}{|x|\,|\theta|} \right\} \quad (55)$$

$$\alpha_8 = \sup_t \sup_{(\tilde{\theta},\dot{\tilde{\theta}})\in\Delta'} \sup_{|x|\neq 0} \left\{ \frac{|(D_\theta G(\theta_d+\tilde{\theta}) - D_\theta G(\theta_d))(x)|}{|x|\,|\theta|} \right\} \quad (56)$$

Assumptions 1, 2, and 3 together imply that α_1 through α_8 are bounded. For convenience, we give names to several combinations of these constants:

$$\gamma_1 = \alpha_1 + \alpha_2 + \alpha_7 + \alpha_8 \quad \gamma_3 = \frac{\alpha_3}{2} + \alpha_6$$
$$\gamma_2 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \quad \gamma_4 = \alpha_4 + \alpha_5 + \alpha_6 \quad (57)$$

It is interesting to note that γ_2, γ_3, and γ_4 are all $O(\delta_2) + O(|\dot{\theta}_d|)$, but that γ_1 is $O(1) + O(|\ddot{\theta}_d|) + O(|\dot{\theta}_d|^2)$. The $O(1)$ contribution is due to α_8, which is the bound on the gravity term.

We are now in position to state the main result of this article.

Theorem 2. Assume that Δ and θ_d are given and that Assumptions 1, 2, and 3 hold. Let $A', B', C' = (B')^T$ be the state space representation of any strictly proper, SPR transfer function. Let K_p' be any positive definite symmetric matrix. Let

$$A = \lambda A' \; B = \lambda B'$$
$$C = \lambda C' \; K_p = \lambda^2 K_p' \quad (58)$$

Then there exists bounded λ^* such that for all $\lambda \geq \lambda^*$, the zero solution of (20), (46) is exponentially stable.

Proof. Along solutions of the nonlinear system (20), (46) the Lyapunov function V given by (25) has the derivative

$$\begin{aligned}\dot{V}^{(NL)} &= (1-b)\langle z, \dot{z}\rangle + \langle K_p\tilde{\theta}, \dot{\tilde{\theta}}\rangle + cM(\theta)(\dot{\tilde{\theta}}, \dot{\tilde{\theta}}) + \\ &\quad b\langle A^{-1}\dot{z}, B\dot{\tilde{\theta}}\rangle \\ &\quad + c\langle K_d\dot{\tilde{\theta}}, \tilde{\theta}\rangle + c\langle A^{-1}\dot{z}, B\tilde{\theta}\rangle + c\langle A^{-1}z, B\dot{\tilde{\theta}}\rangle \\ &\quad + b\langle \ddot{\tilde{\theta}}, B^T A^{-1} z\rangle \\ &\quad + M(\theta)(\ddot{\tilde{\theta}}\dot{\tilde{\theta}}) + \frac{1}{2} D_\theta M(\theta)(\dot{\tilde{\theta}}, \dot{\tilde{\theta}})(\dot{\theta}) \\ &\quad + cM(\theta)(\ddot{\tilde{\theta}}, \tilde{\theta}) + cD_\theta M(\theta)(\dot{\tilde{\theta}}, \tilde{\theta})(\dot{\theta})\end{aligned} \quad (59)$$

where we have assumed symmetry of $[M]$, K_p, and K_d with K_d given by (21). Let $b = \gamma c$ and choose c^*, c^{**}, and c according to (32)–(34). Then, for $(\tilde{\theta}, \dot{\tilde{\theta}}) \in \Delta$ the right-hand side of (59) can be bounded using (38) by

$$\begin{aligned}\dot{V}^{(NL)} &\leq -cV + c\gamma\beta_1\beta_3\gamma_1|z|\,|\tilde{\theta}| + c\gamma\beta_1\beta_3\gamma_2|z|\,|\dot{\tilde{\theta}}| \\ &\quad + c\gamma_1|\tilde{\theta}|^2 + (c\gamma_4 + \gamma_1)|\tilde{\theta}|\,|\dot{\tilde{\theta}}| + \gamma_3|\dot{\tilde{\theta}}|^2\end{aligned}$$

Now let $\overline{\beta}_p'$ be $\overline{\beta}_p$ when $\lambda = 1$ and similarly for all the other constants used in these proofs. Note that the λ dependencies described in (42) still hold and that β_3 and γ_1 through γ_4 do not depend on λ. Let ε be an arbitrary constant between 0 and 1. Define $\varepsilon_1 = (1-\varepsilon)(1-\underline{a}')$. We add and subtract εcV to the right-hand side of (60) to get

$$\dot{V}^{(NL)} \leq -\lambda\varepsilon c'V - \frac{1}{2}\begin{bmatrix}|z|\\|\tilde{\theta}|\\|\dot{\tilde{\theta}}|\end{bmatrix}^T \begin{bmatrix}\varepsilon_1 c'\lambda & -\gamma'\beta_1'\beta_3'\gamma_1 c' & -\gamma'\beta_1'\beta_3'\gamma_2 c'\\ -\gamma'\beta_1'\beta_3'\gamma_1 c' & \varepsilon_1 c'\underline{\sigma}(K_p')\lambda^2 - 2\gamma_1 c'\lambda & -(\gamma_1+\gamma_4 c'\lambda)\\ -\gamma'\beta_1'\beta_3'\gamma_2 c' & -(\gamma_1+\gamma_4 c'\lambda) & \varepsilon_1 c'(\beta_3')^{-2}\lambda - 2\gamma_3\end{bmatrix}\begin{bmatrix}|z|\\|\tilde{\theta}|\\|\dot{\tilde{\theta}}|\end{bmatrix} \tag{61}$$

It is obvious that the matrix on the right-hand side of (61) is positive definite for all λ sufficiently large. Therefore we can choose λ^* as the largest positive solution of

$$0 = \det\begin{bmatrix}\varepsilon_1 c'\lambda^* & -\gamma'\beta_1'\beta_3'\gamma_1 c' & -\gamma'\beta_1'\beta_3'\gamma_2 c'\\ -\gamma'\beta_1'\beta_3'\gamma_1 c' & \varepsilon_1 c'\underline{\sigma}(K_p')(\lambda^*)^2 - 2\gamma_1 c'\lambda^* & -(\gamma_1+\gamma_4 c'\lambda^*)\\ -\gamma'\beta_1'\beta_3'\gamma_2 c' & -(\gamma_1+\gamma_4 c'\lambda^*) & \varepsilon_1 c'(\beta_3')^{-2}\lambda^* - 2\gamma_3\end{bmatrix} \tag{62}$$

Recall that this bound is accurate only when $(\tilde{\theta}, \dot{\tilde{\theta}}) \in \Delta$. We must, therefore, ensure that $(\tilde{\theta}, \dot{\tilde{\theta}}) \in \Delta$ for all time greater than the initial time. Fix $\lambda \geq \lambda^*$ and define a constant v^* such that $V \leq v^*$ implies that $(\tilde{\theta}, \dot{\tilde{\theta}}) \in \Delta$,

$$v^* = \inf_{t\geq 0} \inf_{(\tilde{\theta},\dot{\tilde{\theta}})\notin\Delta} \inf_{z\in R^n} V(z, \tilde{\theta}, \dot{\tilde{\theta}}) \tag{63}$$

where the only time dependence is in θ_d in the terms of V containing $M(\theta) = M(\theta_d + \tilde{\theta})$.

Lemma. If $V(z(0), \tilde{\theta}(0), \dot{\tilde{\theta}}(0)) < v^*$, then the solution of (20), (46) converges exponentially to zero with rate $\lambda\varepsilon c'$.

The proof of this lemma follows standard contradiction arguments. This concludes the proof of the theorem. QED.

5. Conclusions

We have demonstrated the exponential stability of robot controllers consisting of a strictly proper PR controller together with dynamic compensation. This is an extension of our previous work, in which we required that $Z(\infty)$ be positive definite in order to show exponential stability. Furthermore, we have developed bounds on the convergence rate of the control scheme.

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References

[1] B. Paden and B. Riedle, "A Positive-Real Modification of a Class of Nonlinear Controllers for Robot Manipulators." *Proc. American Control Conf.*, Atlanta, GA, USA, June 1988, 77–85.

[2] M. Takegaki and S. Arimoto, "A New Feedback Method for Dynamic Control of Manipulators." *J. Dyn. Sys., Meas. and Cont.*, 102: 119–125.

[3] B. Paden and R. Panja, "Globally Asymptotically Stable 'PD+' Controller for Robot Manipulators." *Int. J. Control*, 47(6), 1697–1712.

[4] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method.* (New York: Springer-Verlag, 1977).

[5] J.J.E. Slotine and W. Li, "On the Adaptive Control of Robot Manipulators." *Int. J. Robotics Research, (6)*3: 49–59, 1987.

[6] J.T. Wen and D. S. Bayard, "Robust Control for Robotic Manipulators, Part I: Non-Adaptive Case." *Int. J. Control, (47)*5: 1361–1385, 1988 .

[7] D.S. Bayard and J.T. Wen, "Robust Control for Robotic Manipulators, Part II: Adaptive Case." *Int. J. Control, (47)*5: 1387–1406, 1988.

[8] N. Sadegh and R. Horowitz, "Stability Analysis of an Adaptive Controller for Robot Manipulators." *Proc. IEEE Int. Conf. on Robotics and Automation*, 3: 1223–1229, March 1987.

[9] S. Arimoto and F. Miyazaki, "Asymptotic Stability of Feedback Control Laws for Robot Manipulators." Paper presented at ASME Winter Annual Meeting, Anaheim, CA, USA, 1986.

[10] B.D.O. Anderson, "A System Theory Criterion for Positive Real Matrices." *J. SIAM Control*, 5(2): 1967.

[11] R.W. Newcomb, *Linear Multiport Synthesis.* (New York: McGraw-Hill, 1966).

[12] B.D.O. Anderson, "A Simplified Viewpoint of Hyperstability." *IEEE Trans. on Aut. Contr.*, June 1968.