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Lyapunov Stability Theory of Nonsmooth Systems

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Abstract—**This paper develops nonsmooth Lyapunov stability theory and LaSalle's invariance principle for a class of nonsmooth Lipschitz continuous Lyapunov functions and absolutely continuous state trajectories. Computable tests based on Filipov's differential inclusion and Clarke's generalized gradient are derived. The primary use of these results is in analyzing the stability of equilibria of differential equations with discontinuous right-hand side such as in nonsmooth dynamic systems or variable structure control.**

I. INTRODUCTION

There are many systems which have nonsmooth dynamics. Examples include systems with Coulomb friction, contact interactions, and variable structure systems where control inputs are allowed to be discontinuous. It is essential to rigorously analyze these systems and address such issues as the existence of equilibria, their stability, and qualitative dynamics. As important as nonsmooth systems are in practice, techniques are still lacking for their analysis. All classical existence theorems for ordinary differential equations require vector fields which are at least Lipschitz continuous. The aforementioned examples, and many others, fail this requirement. With respect to these classical techniques, one cannot even define a solution, much less discuss existence of equilibria and stability.

What is needed is a set of tools which allow the analysis of differential equations with discontinuous right-hand sides. The seminal contribution in this area was made by Filipov [4] who developed a solution concept for differential equations whose right-hand sides were only required to be Lebesgue measurable in the state and time variables. Using this framework, theorems were proved for existence, uniqueness, and continuous dependence on initial conditions. One area missing from this program is the stability analysis of equilibria using nonsmooth Lyapunov functions.

Yoshizawa [14] developed the Lyapunov theory for Lipschitz potential functions, but this work assumed a continuous vector field and smooth trajectories. In his book [6], Filipov studies the equilibria of differential equations with discontinuous right-hand sides, but deals with smooth Lyapunov functions.

There has been work on Lyapunov stability theory in variable structure systems; see [12], [3] for a detailed list of references. In variable structure systems one tries to pick a sliding surface and show that nearby trajectories converge first to this surface, then once on the surface, to an equilibrium point. The control is chosen in such a way that the dynamics of the closed-loop system never pass through the surface and all analysis can be done from a single side of the switching surface, and then once the trajectory is on the switching surface and smooth dynamics are restored, ordinary Lyapunov theory is adequate. As a result of this piecewise approach, the nonsmoothness of the trajectory is unimportant and differentiable Lyapunov functions suffice in many cases. For our purposes, the "kinks" are an essential part of the dynamics, and this one-sided analysis is inadequate.

The nonsmooth Lyapunov analysis of equilibria is present in the differential inclusions literature [1], [2], [7], of which Filipov's differential inclusion is a special example. Theorems are developed which rely on the notions of set valued maps and derivatives. An alternative formulation using Dini derivatives was developed by Roxin [11]. Although very general, both these techniques lack a calculus for computing derivatives of nonsmooth Lyapunov functions. We resolve this problem by exploiting the additional structure of Filipov's solution to develop more useful tools in this context. Additionally, the motivation for studying general differential inclusions is different in that the focus is on integral funnels, in which one searches for monotone trajectories which tend to the equilibrium.

The purpose of this work is to provide some tools of generalized Lyapunov analysis with which the stability properties of nonsmooth dynamic systems can be determined. These tools will turn out to be computable and easily applicable making nonsmooth Lyapunov analysis no more difficult than its smooth counterpart. Specifically, our contribution is weakening the restriction of differentiability to a broad class of nonsmooth Lipschitz continuous Lyapunov functions; the trajectories are only required to be absolutely continuous. This is important because we will show there are nonsmooth dynamic systems whose equilibria cannot be proved stable using continuously differentiable Lyapunov theory. In addition, nonsmooth Lyapunov functions are natural for nonsmooth dynamic systems.

II. MATHEMATICAL FRAMEWORK

In this section we review the Filipov solution concept for differential equations with discontinuous right-hand sides, the nonsmooth analysis of Clarke's generalized gradient, and develop a connection via a new chain rule for differentiating regular functions along Filipov solution trajectories.

Filipov Solutions

We consider the vector differential equation

$$\dot{x} = f(x, t) \tag{1}$$

where $f\colon R^n \times R \to R^n$ is measurable and essentially locally bounded. We must first define what it means to be a solution of this equation.

Definition 2.1 (Filipov): A vector function $x(\cdot)$ is called a solution of (1) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$

$$\dot{x} \in K[f](x, t) \tag{2}$$

where

$$K[f](x, t) \equiv \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{co} f(B(x, \delta) - N, t) \tag{3}$$

and $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero. An equivalent definition [5], [10] is: there exists $N_f \subset R^m$, $\mu N_f = 0$ such that for all $N \subset R^m$, $\mu N = 0$

$$K[f](x) \equiv \overline{co}\{\lim f(x_i) \mid x_i \to x,\ x_i \notin N_f \cup N\}. \tag{4}$$

The content of Filipov's solution is that the tangent vector to a solution, where it exists, must lie in the convex closure of the limiting values of the vector field in progressively smaller neighborhoods around the solution point. It is important in the above definition that we discard sets of measure zero. This technical detail allows solutions to be defined at points even where the vector field itself is not defined, such as at the interface of two regions in a piecewise

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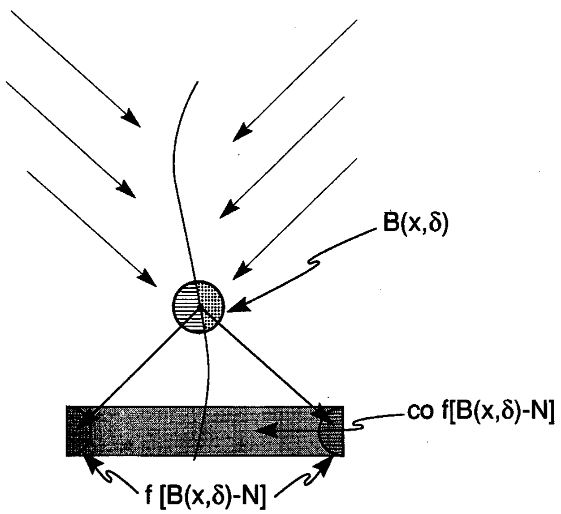

Fig. 1. The limiting procedure used to calculate $K[f](x, t)$.

defined vector field. Fig. 1 helps illustrate the limiting procedure used to define $K[f](x, t)$. This figure shows the vector images of a small neighborhood around the base point x. The interface is a neglected set of measure zero where the vector field is not defined. The set $K[f](x, t)$ reduces to the convex hull of two limit vectors as the neighborhood becomes vanishingly small.

Example: A nontrivial example of a nonsmooth dynamic system is the following defined for almost all $x \in R^2$

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla \|x\|_1 \tag{5}$$

where $\|x\|_1 = |x_1| + |x_2|$. The gradient in (5) is not defined on the x_1 and x_2 axes, sets of measure zero. Off the axes, the Filipov set $K[f](x)$ is a singleton

$$K[f](x, t) = \begin{cases} \{(-1, +1)^T\} & \text{for } x \in \text{quadrant 1} \\ \{(-1, -1)^T\} & \text{for } x \in \text{quadrant 2} \\ \{(+1, -1)^T\} & \text{for } x \in \text{quadrant 3} \\ \{(+1, +1)^T\} & \text{for } x \in \text{quadrant 4.} \end{cases} \tag{6}$$

On the x_1 and x_2 axes, $K[f](x, t)$ is the convex hull of each of the vectors in (6) corresponding to the quadrants which the point x borders. For example, on the positive x_1 axis $K[f](x) = \overline{co}\{(-1, 1)^T, (1, 1)^T\}$. Any Filipov solution to the differential equation (5) traces a square. observe that the trajectories in this example move along level sets of $\|x\|_1$, a natural (nonsmooth) Lyapunov function for this system.

In most applications the calculation of $K[f](x, t)$ involves functions $f(x, t)$ expressed as sums, products, and compositions of other functions. Hence a calculus is needed for computing the Filipov set. This calculus was derived in our earlier work [10].

We do not require Lipschitz continuity of $f(x, t)$ in x or t and therefore cannot in general expect uniqueness and continuous dependence on initial conditions, although we will assume both in what follows. Readers interested in more detailed discussions are referred to the original literature [4], [5].

Generalized Gradients

As shown in the example above, nonsmooth Lyapunov functions arise naturally in the stability theory of differential equations with discontinuous right-hand sides. In the application of the machinery of nonsmooth analysis, Clarke's generalized gradient [2] is particularly useful in simplifying proofs.

Definition 2.2 (Clarke's Generalized Gradient): For a locally Lipschitz function $V\colon R^n \times R \to R$ define the generalized gradient of V at (x, t) by

$$\partial V(x, t) = \overline{co}\{\lim \nabla V(x, t) \mid (x_i, t_i) \to (x, t), (x_i, t_i) \notin \Omega_V\} \tag{7}$$

where Ω_V is the set of measure zero where the gradient of V is not defined.

The gradient ∇ includes the derivative with respect to time $(\partial/\partial t)$. In this definition, Lipschitz means Lipschitz in (x, t) (discontinuities in t are not allowed). For notational brevity, if a function $V(x, t)$ has no explicit t dependence, we shall adopt the convention of dropping the last component of ∂V which is identically zero.

One way to view the generalized gradient at a point x is a set valued map equal to the convex closure of the limiting gradients near x. For example, the function

$$V(x) = |x|, \quad x \in R \tag{8}$$

has a generalized gradient which equals

$$\begin{aligned} \partial V(x) &= \{-1\} \quad x \in R^- \\ &= \{+1\} \quad x \in R^+ \\ &= [-1, 1] \quad x = 0. \end{aligned} \tag{9}$$

The next lemma states that the generalized directional derivative, which is defined presently, is the support function in the sense of convex analysis for the generalized gradient. Both the definition and result are due to Clarke [2].

Definition 2.3: The generalized directional derivative is defined

$$f^\circ(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{f(y + tv) - f(y)}{t}. \tag{10}$$

Lemma 2.1: Let f be Lipschitz near x, then

$$f^\circ(x; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial f(x)\}. \tag{11}$$

Our chain rule will be for a useful class of functions, called regular functions [2].

Definition 2.4: $f(x, t)\colon R^m \times R \to R$ is called regular if

1) for all v, the usual one-sided directional derivative $f'(x; v)$ exists,

2) for all v, $f'(x; v) = f^\circ(x; v)$.

Examples of regular functions include smooth functions and functions which can be written as the pointwise maximum of a set of smooth functions, such as $\|x\|_1$. When $x(t)$ is a Filipov solution to $\dot{x} = f(x, t)$ and $V(x, t)$ is a regular function, then $(d/dt)V(x(t), t)$ can be expressed in terms of ∂V and $K[f](x, t)$.

Theorem 2.2 (Chain Rule): Let $x(\cdot)$ be a Filipov solution to $\dot{x} = f(x, t)$ on an interval containing t and $V\colon R^n \times R \to R$ be a Lipschitz and in addition, regular function. Then $V(x(t), t)$ is absolutely continuous, $(d/dt)V(x(t), t)$ exists almost everywhere and

$$\frac{d}{dt} V(x(t), t) \in^{\text{a.e.}} \dot{\tilde{V}}(x, t) \tag{12}$$

where

$$\dot{\tilde{V}}(x, t) := \bigcap_{\xi \in \partial V(x(t), t)} \xi^T \begin{pmatrix} K[f](x(t), t) \\ 1 \end{pmatrix}. \tag{13}$$

Proof: That $V(x(t), t)$ is absolutely continuous and $(d/dt)V(x(t), t)$ exists almost everywhere is a consequence of the fact that the composition of a Lipschitz function, $V(\cdot)$, with an absolutely continuous function, $(x(t), t)$, is absolutely continuous (see, for example, [9]). At a point where $x(t)$ and $V(x(t), t)$ are both differentiable (this is true almost everywhere)

$$\begin{aligned}
&\frac{d}{dt}V(x(t), t) && (14)\\
&= \lim_{h\downarrow 0}\frac{V(x(t+h), t+h) - V(x(t), t)}{h} && (15)\\
&= \lim_{h\downarrow 0}\frac{V(x(t)+\dot{x}h, t+h) + o(h) - V(x(t), t)}{h} && (16)\\
&= V'((x, t)^T; (\dot{x}, 1)^T) && (17)\\
&= V^{\circ}((x, t)^T; (\dot{x}, 1)^T) \quad \text{by regularity} && (18)\\
&= \max\{\langle\xi|(\dot{x}, 1)^T\rangle | \xi \in \partial V(x, t)\}. && (19)
\end{aligned}$$

a similar argument shows

$$\begin{aligned}
&\lim_{h\downarrow 0}\frac{V(x(t+h), t+h) - V(x(t), t)}{h} && (20)\\
&\quad = \min\{\langle\xi|(\dot{x}, 1)^T\rangle | \xi \in \partial V(x, t). && (21)
\end{aligned}$$

Therefore

$$\frac{d}{dt}V(x(t), t) = \langle\xi \mid (\dot{x}, 1)^T\rangle \quad \forall\xi \in \partial V(x, t). \tag{22}$$

Now, x is a Filipov solution so that

$$\dot{x}(t) \in K[f](x(t)), \quad \text{a.e.} \tag{23}$$

Thus almost everywhere, $\dot{V} = \xi^T\begin{pmatrix}\eta\\1\end{pmatrix}$ for all $\xi \in \partial V(x, t)$ and some $\eta \in K[f](x, t)$. Equivalently

$$\dot{V} \in^{\text{a.e.}} \dot{\tilde{V}} \equiv \bigcap_{\xi\in\partial V(x(t), t)} \xi^T\begin{pmatrix}K[f](x(t), t)\\1\end{pmatrix}. \qquad \square \tag{24}$$

Example: Let $V(x) = \|x\|_1$ and $f(x)$ as in the example, (5), above. Let $x(t)$ be the solution passing through $(1, 0)^T$ at time $t = 0$. We have

$$\partial V(x(0), 0) = \begin{pmatrix}1\\ [-1, +1]\end{pmatrix} \tag{25}$$

(where we have employed our convention of dropping the last term in $\partial V(x, t)$ when V is independent of time) and

$$K[f](x(0), 0) = \begin{pmatrix}[-1, +1]\\ 1\end{pmatrix}. \tag{26}$$

Let $(1, \xi_2)$, $\xi_2 \in [-1, 1]$ be an arbitrary element of $\partial V(x(0), 0)$ then

$$\begin{bmatrix}1\\ \xi_2\end{bmatrix}K[f](x(0), 0) = \xi_2 + [-1, +1] = [\xi_2 - 1, \xi_2 + 1] \tag{27}$$

implies

$$\dot{\tilde{V}}(x(0), 0) = \bigcap_{\xi_2\in[-1, 1]} [\xi_2 - 1, \xi_2 + 1] = 0. \tag{28}$$

This does not guarantee that $\dot{V} = 0$ (or even exists) at this point, but $\dot{\tilde{V}}(x) \leq 0$ by which we mean $v < 0$ for all $v \in \dot{\tilde{V}}$, does guarantee stability. The theorems of the next section formalize this.

III. Stability Theorems

In this section we state two existing Lyapunov theorems (uniform stability and uniform asymptotic stability) in terms of the set valued map $\dot{\tilde{V}}$. The proofs are omitted because they are identical to their smooth counterparts except for some relations holding "almost everywhere" instead of everywhere. See Khalil and Vidyasagar [8], [13] for the smooth versions. Proofs exist for the other versions of stability, such as nonuniform, global, etc., but for the sake of brevity we have omitted them. An application of this theorem is made, and the stability of the spring-mass-coulomb friction system is proved.

Theorem 3.1: Let $\dot{x} = f(x, t)$ be essentially locally bounded and $0 \in K[f](0, t)$ in a region $Q \supset \{x \in R^n \mid \|x\| < r\} \times \{t | t_0 \leq t < \infty\}$. Also, let $V\colon R^n \times R \to R$ be a regular function satisfying

$$V(0, t) = 0 \tag{29}$$

and

$$0 < V_1(\|x\|) \leq V(x, t) \leq V_2(\|x\|) \quad \text{for } x \neq 0 \tag{30}$$

in Q for some V_1, $V_2 \in$ class $\mathcal{K}$. (See [8] for a definition of class $\mathcal{K}$ functions.) Then

1) $\dot{\tilde{V}}(x, t) \leq 0$ in Q implies $x(t) \equiv 0$ is a uniformly stable solution.
2) If in addition, there exists a class $\mathcal{K}$ functions $\omega(\cdot)$ in Q with the property

$$\dot{\tilde{V}}(x, t) \leq -\omega(x) < 0 \tag{31}$$

then the solution $x(t) \equiv 0$ is uniformly asymptotically stable.

Example ([6], [10]): Let $R(x, t)$ be a matrix which is continuous and uniformly positive definite when symmetrized and

$$\dot{x} = -R(x(t), t)\,\nabla\|x\|_1. \tag{32}$$

Then $x(t) \equiv 0$ is asymptotically stable.

Proof: Choose $V(x) = \|x\|_1$. Then

$$\dot{\tilde{V}}(x, t) = \bigcap_{\xi\in\partial\|x\|_1} \xi^T K[-R(x, t)\,\nabla\|x\|_1](x, t) \tag{33}$$

(since V is time independent) and by the calculus for K [10], we have

$$\dot{\tilde{V}}(x, t) = \bigcap_{\xi\in\partial\|x\|_1} -\xi^T R(x, t)\partial\|x\|_1. \tag{34}$$

Since R is uniformly positive definite when symmetrized, there exists $\rho > 0$ such that

$$\xi^T R(x, t)\xi \geq \rho\|\xi\|^2. \tag{35}$$

$\partial\|x\|_1$ is convex so

$$\xi_0(x, t) = \arg\min_{\xi\in\partial\|x\|_1} \xi^T\left(\frac{R(x, t) + R^T(x, t)}{2}\right)\xi \tag{36}$$

satisfies

$$\begin{aligned}
&\xi^T\left(\frac{R(x, t) + R^T(x, t)}{2}\right)\xi_0(x, t)\\
&\quad\geq \xi_0(x, t)^T\left(\frac{R(x, t) + R^T(x, t)}{2}\right)\xi_0(x, t)\\
&\quad\geq \rho\|\xi_0(x, t)\|^2 && (37)
\end{aligned}$$

for all $\xi \in \partial\|x\|_1$. Equation (37) is a consequence of the fact that for a convex domain D and a smooth function f at the point $x_0 = \arg\min f(x)$, the point x_0 satisfies

$$\nabla f(x_0)\cdot(y - x_0) \geq 0 \quad \text{for all } y \in D. \tag{38}$$

This shows

$$\dot{\tilde{V}}(x, t) \leq -\rho \|\xi_0(x, t)\|^2. \tag{39}$$

At $x = 0$, $\partial \|x\|_1 = [-1, +1]^n$. Since $\|x\|_1$ is convex, $\partial \|x\|_1|_{x \neq 0} \cap (-1, +1)^n = \varnothing$ (this follows from [2, Proposition 2.2.9]). Thus

$$\underset{\xi \in \partial \|x\|_1|_{x \neq 0}}{\arg\min} \|\xi\|^2 \geq 1 \tag{40}$$

so $\dot{\tilde{V}} \leq -\rho$ almost everywhere. This implies $x(t) \to 0$ asymptotically (converges in finite time in fact by the finite time stability theorem in [10]).

The proof just demonstrated shows, with minor modifications, the example in the last section where

$$R(x, t) \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{41}$$

has zero as a stable equilibrium point. We now argue that this equilibrium cannot be shown stable by any smooth time independent Lyapunov function.

The trajectories of this dynamic system are squares. Any nonincreasing function along a closed curve must be a constant, therefore the level sets of any proposed Lyapunov function must be squares. The level sets of any smooth function with nonzero gradients are smooth precluding the possibility of any smooth Lyapunov function for this example. A similar argument also shows the impossibility of finding a continuously differential time dependent Lyapunov function.

We now prove a nonsmooth version of LaSalle's theorem.

Theorem 3.2 (LaSalle): Let Ω be a compact set such that every Filipov solution to the autonomous system $\dot{x} = f(x)$, $x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V\colon \Omega \to R$ be a time independent regular function such that $v \leq 0$ for all $v \in \dot{\tilde{V}}$ (if $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega \,|\, 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set, M, in the closure of S.

Proof: Let $x(t)$ be a Filipov solution starting in Ω. Since $V(x(t))$ is absolutely continuous, $\dot{\tilde{V}}$ is bounded below zero, and V is bounded above zero, $V(x)$ tends to a constant, a, as $t \to \infty$. Uniqueness of trajectories implies continuous dependence on initial conditions so the positive limit set, L^+, of $x(t)$ is an invariant set. Moreover, $L^+ \subset \Omega$ because Ω is closed. By continuity of V, $V(p) = a$ for all $p \in L^+$. Since L^+ is invariant we have $\dot{V} = 0$ on L^+, hence by Theorem 2.2, $0 \in \dot{\tilde{V}}(x)$, a.e. in t. It follows by the lemma below that $L^+ \subset \overline{S}$. Since L^+ is contained in the largest invariant set in $\overline{S}$ the theorem is proved. □

Lemma 3.3: Under the conditions of the theorem above

$$L^+ \subset \overline{S}. \tag{42}$$

Proof: Let $x_0 \in L^+$ and $s(t, x_0, t_0)$ be the solution to $\dot{x} = f(x)$, $x_0 = x(t_0)$. Since $s(t, x_0, t_0)$ lies in L^+, $(d/dt)V(s(t, x_0, t_0)) = 0$. Since $\dot{V}(x(t)) \in \dot{\tilde{V}}(x)$ a.e., in t, we have $s(t, x_0, t_0) \in S$ a.e. Thus $s(t, x_0, t_0) \in S$ for some $t \in [t_0, t_0 + \delta)$ and all $\delta > 0$. Since $s(t, x_0, t_0)$ is continuous (absolutely) we have that there are points in S arbitrarily close to x_0 implying $x_0 \in \overline{S}$. Hence $L^+ \subset \overline{S}$. □

The following theorem is a special case of a viability result in the study of differential inclusions (see [1, p. 180] for a general statement and complete proof).

Theorem 3.4: If M is an invariant set in a smooth k-dimensional manifold S, then

$$T_m S \cap K[f](m) \neq \varnothing \tag{43}$$

for all $m \in M$.

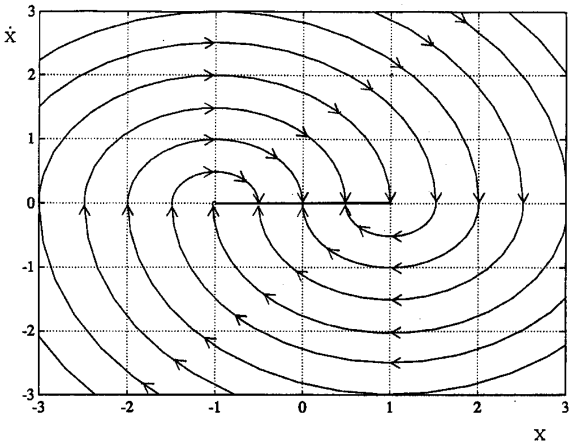

Fig. 2. The phase portrait of $m\ddot{x} + b\,\mathrm{sgn}\,(\dot{x}) + kx = 0$ with $m = b = k = 1$.

Example: Consider a harmonic oscillator with Coulomb friction

$$m\ddot{x} + b\,\mathrm{sgn}\,(\dot{x}) + kx = 0 \tag{44}$$

or equivalently

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\,\mathrm{sgn}\,(\dot{x}) - \frac{k}{m}x \end{bmatrix} = f(x, \dot{x}). \tag{45}$$

Choose the (smooth, time independent) Lyapunov function

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \tag{46}$$

Then

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(x, \dot{x})} \xi^T K \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\,\mathrm{sgn}\,(\dot{x}) - \frac{k}{m}x \end{bmatrix}. \tag{47}$$

Since V is smooth

$$\begin{aligned} \dot{\tilde{V}} &= \nabla V^T K \begin{bmatrix} \dot{x} \\ -\frac{b}{m}\,\mathrm{sgn}\,(\dot{x}) - \frac{k}{m}x \end{bmatrix} \\ &\subset \begin{bmatrix} kx \\ m\dot{x} \end{bmatrix}^T \begin{bmatrix} \dot{x} \\ -\frac{b}{m}K[\mathrm{sgn}\,(\dot{x})] - \frac{k}{m}x \end{bmatrix} \\ &= -b\dot{x}\,\mathrm{SGN}\,(\dot{x}) \\ &= -b|\dot{x}| \end{aligned} \tag{48}$$

where

$$\mathrm{SGN}\,(x) = \begin{cases} -1 & x < 0 \\ [-1, 1] & x = 0 \\ 1 & x > 0. \end{cases} \tag{49}$$

This implies $(x, \dot{x})$ approaches the largest invariant set in

$$\overline{S} = \mathrm{cl}\,(\{(x, \dot{x}) \mid 0 \in \dot{\tilde{V}}(x, \dot{x})\}). \tag{50}$$

By Theorem 3.4

$$T_m\overline{S} \cap K[f](m) \neq \varnothing \tag{51}$$

for all $m \in M$. We can compute

$$T_m\overline{S} = \mathrm{Span}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{52}$$

and

$$K[f](m) = \begin{bmatrix} 0 \\ -\frac{b}{m}[-1, 1] - \frac{k}{m}x \end{bmatrix}. \tag{53}$$

The intersection of (52) and (53) is $[0, 0]^T$ provided $kx \in [-b, b]$ implying the largest invariant set is contained in $([-(b/k), (b/k)], 0)$. See Fig. 2 for a phase portrait.

IV. CONCLUSION

In this paper we have extended basic Lyapunov stability theorems to the nonsmooth case using the Filipov solution concept and Clarke's generalized gradient. The result is a theory applicable to systems with switches for which natural Lyapunov functions are often only piecewise smooth. This machinery should find application in variable structure control theory, the analysis and control of mechanical systems, and the analysis of pulse width modulated control systems.

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Risk-Sensitive Estimation and a Differential Game

Ravi N. Banavar and Jason L. Speyer

***Abstract*— A large deviation result is employed to solve the state estimation problem of a continuous time Gauss–Markov system with an exponential cost. The exponential cost is the expected value of an exponential function of the state estimation-error. A scalar θ appearing in the cost, termed as the risk factor, determines the penalty on the higher order moments of the error. In contrast to the minimum variance estimate, penalty on large deviations of the estimation error is possible.**

I. INTRODUCTION

The expected value of an exponential function as a performance measure—henceforth referred to as exponential cost (EC)—was initially proposed by Jacobson [11]. The cost is employed as an optimality criterion for a Gauss–Markov system with perfect knowledge of the systems states. The EC includes as a special case the linear quadratic Gaussian (LQG) cost. The performance measure was later examined, for the case of imperfect state observation by [17], [20], [2], [12]. In [21], Whittle views the cost as a risk-sensitive criterion and lends a desirable certainty equivalence principle to the problem. An application to missile guidance is demonstrated in [18].

Interest in the exponential cost problem was revived when the solutions obtained from it were linked to those of the H_∞ problem [8], [4]. In his initial paper, Jacobson [11] demonstrates the link between the EC and a differential game. Solutions to the H_∞ problem for time-varying systems is obtainable through a differential game approach [1], [15], [13]. The differential game commonality links the stochastic problem with the EC to the deterministic problem with a worst-case measure (H_∞). Most of the existing results on the exponential cost problem assume an exact equivalence to optimizing a quadratic performance measure. More recently, the problem has been re-examined from a large deviation perspective. Whittle [24], [25] states a risk-sensitive maximum principle derived from a result in large deviation theory. Nonlinear extensions to the same are found in [6]. Recasting the problem into a large deviation framework highlights the suboptimal nature of the solution obtained through a quadratic kernel optimization. This is also in consonance with the deterministic H_∞ theory where suboptimal solutions are obtained [4].

This paper examines the continuous time version of the estimation problem on a finite time interval. Speyer *et al.* [19] have considered the discrete-time version of the problem. As stated in [19], the estimation problem in this stochastic setting is particularly interesting since the cost function is a curious exception to the family of functions proposed by Sherman [16], [9] that yield a conditional mean as the optimal estimator. Here, the performance measure is recast into a large deviation framework and a suboptimal solution is obtained by invoking a result in large deviation. The approach is similar to Whittle's [25]. The two main theorems are stated and proved in Section II.

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