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Exact-output tracking theory for systems with parameter jumps

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We consider the exact output tracking problem for systems with parameter jumps. Necessary and sufficient conditions are derived for the elimination of switching-introduced output transient. Previous works have studied this problem by developing a regulator that maintains exact tracking through parameter jumps (switches). Such techniques are, however, only applicable to minimum-phase systems. In contrast, our approach is applicable to non-minimum-phase systems and it obtains bounded but possibly non-causal solutions. If the reference trajectories are generated by an exosystem, then we develop an exact-tracking controller in a feedback form. As in standard regulator theory, we obtain a linear map from the states of the exosystem to the desired system state which is defined via a matrix differential equation. The constant solution of this differential equation provides asymptotic tracking, and coincides with the feedback law used in standard regulator theory. The obtained results are applied to a simple flexible manipulator with jumps in the pay-load mass.

1. Introduction

We study the exact-output tracking of systems that are described by

$$\left.\begin{aligned} \dot{x}(t) &= A[k(t)]x(t) + B[k(t)]u(t) \\ y(t) &= C[k(t)]x(t) \end{aligned}\right\} \tag{1}$$

where $x \in \mathbb{R}^n$, with the same number of inputs as outputs $u(t), y(t) \in \mathbb{R}^p$. The system matrices $A(k)$, $B(k)$ and $C(k)$ are constant over the time intervals I_k, where k belongs to a finite index set $\mathscr{K} \triangleq [0, \dots, N]$, and the parameter change (switch) occurs at times $t = t_1, t_2, \dots, t_N$ (see Fig. 1). Here, the switching times are known, in contrast to systems where the switches may be signal-driven.

For constant linear systems, asymptotic output-tracking problems have received much attention in the past. In particular, the regulator theory (Francis 1977, Basile and Marro 1992, Wonham 1985) provides a general framework in which the asymptotic output tracking can be solved when the reference trajectory is generated through a linear exosystem. In the presence of switches in the system, one technique for achieving output regulation is to switch the regulator. Note that regulation can be recovered between two consecutive switches (due to asymptotic properties), especially if the switching occurs far apart in time. However, this technique also tends to induce transients in the output during the switches.

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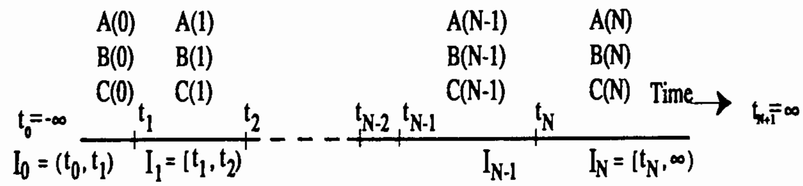

Figure 1. The switching times.

In order to eliminate these switching-caused transients, a regulation scheme that maintains exact trajectory tracking across system switches must be used. This fairly new problem has been studied by Marro and Piazzi (1993) for minimum-phase systems. In this work, a feedforward action is used in conjunction with the feedback defined by the regulator to cancel the output transients across the switches.

We propose an alternative approach for exact output-tracking of switched systems, which is also applicable to non-minimum-phase systems. In the non-minimum-phase case, a bounded non-causal solution is obtained (Devasia *et al.* 1996) that requires preknowledge of the reference trajectory and of all the switching times. Such exact tracking schemes based on non-causal schemes is useful in problems like aircraft guidance (Meyer *et al.* 1995, Hunt *et al.* 1996).

We present necessary and sufficient conditions for the solvability of the inversion problem for linear systems with switches; the inverse is used to track the desired output. We consider two kinds of desired output trajectory: a single pre-specified trajectory, or one belonging to a class of outputs generated by a given linear exo-system, that could undergo parameter changes as well. In this latter case, we obtain the solution in a time-varying feedback form, where the feedback matrix satisfies a matrix ordinary differential equation. The equilibrium solution of this differential equation solves the asymptotic output tracking problem, and coincides with the feedback matrix resulting from the standard regulator. This establishes an interesting connection between our approach and the traditional regulator theory.

The paper is organized as follows: in § 2 the exact tracking of a single prescribed output trajectory is considered, and necessary and sufficient conditions are presented. A geometric version of the obtained conditions is also provided. In § 3 the case of reference trajectory obtained through a linear exosystem will be treated. The conditions of the previous section when rearranged establish a close relationship with the traditional theory of output regulation. Section 4 focuses on the additional problem of stability of the closed loop system. Finally, § 5 presents the application of the developed theory to a simple non-minimum-phase switched system, given by a flexible beam subjected to step variation of the pay-load mass. Conclusions end the paper.

2. Tracking a prescribed output trajectory

Below we formulate the exact tracking problem for a prescribed output trajectory, and establish necessary and sufficient conditions for its solvability. Geometric interpretations of these conditions are also provided.

2.1. *The inversion problem*

Given a desired output trajectory y_d, find a pair of state and input trajectories x_d and u_d such that:

(*a*) x_d and u_d satisfy the system (1):

$$\dot{x}(t) = A[k(t)]x_d(t) + B[k(t)]u_d(t), \quad \forall t \in (-\infty, \infty) \tag{2}$$

(*b*) exact output tracking is achieved (even across switches):

$$y_d(t) = C[k(t)]x_d(t), \quad \forall t \in (-\infty, \infty)$$

(*c*) and the inputs and state are bounded:

$$\|x_d(\cdot)\|_\infty < \infty$$
$$\|u_d(\cdot)\|_\infty < \infty$$

2.2. *Using the inverse for exact-output tracking*

The existence of an inverse (u_d, x_d) implies that there are input-state trajectories that yield the desired output; exact output tracking is easily achieved by stabilizing the desired state trajectory. For example, use u_d as feedforward and use the error $x - x_d$ for feedback (see Fig. 2). Stabilization is not the central issue in this paper, and any scheme for feedback design can be used. For example, given $(A(k), B(k))$ controllable for all k, the system may be stabilized through pole placement with all the closed-loop poles in the same locations for all k.

Note that typically the initial conditions of the system are different from the initial conditions of the desired state trajectory leading to initial transient errors typical of all tracking controllers. However, once the desired level of tracking has been achieved (due to an exponential reduction in error) our technique will maintain tracking across parameter switches. In contrast, switching standard regulators when the system parameters change will cause transient errors at the switching instants; tracking will not be maintained across switches.

2.3. *Exact-tracking maintaining input*

We will assume the following.

Assumption 1: *The system $A(k)$, $B(k)$ and $C(k)$ has a well defined vector relative degree (Isidori 1989) for each $k \in \mathcal{K}$.*

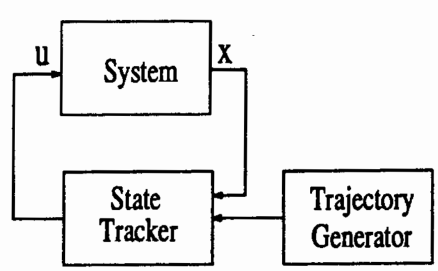

Figure 2. The control scheme.

Then we can find a coordinate transformation $\hat{Q}_k$ such that (Isidori 1989)

$$\begin{bmatrix} Y_k(t) \\ z_k(t) \end{bmatrix} = \hat{Q}_k x(t) = \begin{bmatrix} C_k^* \\ Z_k \end{bmatrix} x(t)$$

where

$$Y_k(t)] \left[y_1, \dot{y}_1, \ldots, \frac{\mathrm{d}^{(r_{k,1}-1)}}{\mathrm{d}t^{(r_{k,1}-1)}} y_1, y_2, \dot{y}_2, \ldots, \frac{\mathrm{d}^{(r_{k,2}-1)}}{\mathrm{d}t^{(r_{k,2}-1)}} y_2, \ldots, y_p, \dot{y}_p, \ldots, \frac{\mathrm{d}^{(r_{k,p}-1)}}{\mathrm{d}t^{(r_{k,p}-1)}} y_p \right]'$$

and $r_k = [r_{k,1} \; r_{k,2} \; \ldots \; r_{k,p}]$ is the system's vector relative degree for $t_k \leqslant t \leqslant t_{k+1}$ (where we assume $t_0 = -\infty$). Here C_k^* maps the system states into the outputs and its time derivatives and Z_p maps the states into the internal dynamics z_k.

Note that a necessary condition for exact output tracking in the interval I_k is that the system state at time t_k is such that

$$Y_k(t_k) = Y_{k,\mathrm{d}}(t_k)$$

In addition, to maintain exact tracking we need to ensure that

$$\dot{Y}_k(t) = \dot{Y}_{k,\mathrm{d}}(t), \quad \forall t \in [t_k, t_{k+1})$$

Let

$$y_\mathrm{d}^{(r_k)} = \left[\frac{\mathrm{d}^{(r_{k,1})}}{\mathrm{d}t^{(r_{k,1})}} y_{\mathrm{d},1}, \frac{\mathrm{d}^{(r_{k,2})}}{\mathrm{d}t^{(r_{k,2})}} y_{\mathrm{d},2}, \ldots, \frac{\mathrm{d}^{(r_{k,p})}}{\mathrm{d}t^{(r_{k,p})}} y_{\mathrm{d},p} \right]'$$

then we can find the following unique control law (provided that Assumption 1 is satisfied) (Isidori 1989):

$$u_\mathrm{d}(t) = F_k x(t) + G_k y_\mathrm{d}^{(r_k)}(t) \tag{3}$$

such that the time derivative of the output is the same as that of the desired output trajectory y_d (this is also a necessary condition for exact output tracking). This exact-tracking control is completely determined by the state $x(t_k)$, and by the desired output along with its derivatives up to the order r_k.

Substituting the control law (3) into (2) we obtain for $t \in I_k$:

$$\dot{x}(t) = A_y(k)x(t) + B_y(k)y_\mathrm{d}^{(r_k)}(t) \tag{4}$$

where $A_y(k) = A(k) + B(k)F_k$, $B_y(k) = B(k)G_k$. In the transformed coordinates the system equations for $t \in I_k$ are of the form

$$\left.\begin{aligned} \dot{Y}_k(t) &= \dot{Y}_{k,\mathrm{d}}(t) \\ \dot{z}_k(t) &= A_z(k)z_k(t) + A_{z,y}(k)Y_k(t) + B_z(k)y_\mathrm{d}^{(r_k)}(t) \end{aligned}\right\} \tag{5}$$

Our objective is to define under which conditions it is possible to define a feasible state trajectory $x_\mathrm{d}(t)$ such that exact trajectory tracking is preserved through all the time intervals. There are two main hurdles. Firstly, the existence of at least a state trajectory which maintains exact output tracking needs to be determined. This depends on compatibility of the desired output with the given system. Secondly, the state trajectories need to be bounded. In systems with unstable internal dynamics (non-minimum-phase systems) generic solutions tend to be unbounded. In this case, we need to establish additional conditions for the existence of bounded solutions to the internal dynamics.

In this paper we restrict ourselves to the case where: $Y_{k,\mathrm{d}}$, the output along with its time derivatives, has a compact support $[T_i, T_f] \in (-\infty, \infty)$; (*b*) the switching occurs within this compact set; and (*c*) the internal dynamics are hyperbolic before T_i and after T_f. More formally, we have the following assumption.

Assumption 2: *The desired output trajectory $y_\mathrm{d}(\cdot)$ and its time derivatives are bounded and have compact support $S(y_\mathrm{d}) = [T_i, T_f]$. The switching in the system parameters are at fixed times $t_k \in (T_i, T_f)$ for every $k \in [1, \dots, N]$.*

Assumption 3: *The system* (1) *has hyperbolic internal dynamics, i.e. the eigenvalues of $A_z(k)$ have non-zero real parts (no centres) for $k = 0$ and $k = N$. This is equivalent to requiring that the original system* (1) *have no zeros which lie on the imaginary axis (Isidori* 1989) *for $k = 0$ and $k = N$.*

The last assumption implies the existence of transformations Q_0 and Q_N such that the system state can be partitioned into

$$\begin{bmatrix} Y_k(t) \\ z_{sk}(t) \\ z_{uk}(t) \end{bmatrix} = Q_k x(t) = \begin{bmatrix} C_k^* \\ Z_{sk} \\ Z_{uk} \end{bmatrix} x(t) \tag{6}$$

where z_{sk} and z_{uk} are the coordinates for the stable and the unstable subspaces of the system's internal dynamics.

2.4. *Notations*

Towards establishing conditions for the existence of solutions to the exact tracking problem for a prescribed output, we first study the dynamic evolution of the system for a given initial condition.

Given an initial condition in an interval I_k, the system's evolution for $t_k \leqslant t \leqslant t_{k+1}$ is described by

$$x_\mathrm{d}(t) = \exp\left[A_y(k)(t - t_k)\right]\, x_\mathrm{d}(t_k) + \int_{t_k}^{t} \exp\left[A_y(k)(t-\tau)\right]\, B_y(k) y_\mathrm{d}^{(r_k)}(\tau)\, \mathrm{d}\tau$$

In a more compact form

$$x_\mathrm{d}(t) = \Phi_k(t, t_k) x_d(t_k) + h_k(t, t_k)$$

where

$$\Phi_k(t, t_k) = \exp\, \left[A_y(k)(t - t_k)\right]$$

and

$$h_k(t, t_k) = \int_{t_k}^{t} \exp\left[A_y(k)(t-\tau)\right] B_y(k) y_\mathrm{d}^{(r_k)}(\tau)\, \mathrm{d}\tau$$

The above equations describe the flow in an interval where the system does not undergo switches. To obtain a representation of the system state in terms of an initial state that does not belong to the same interval, we define flow compositions as follows:

$$\Psi_{k,i}(t, t_i) = \Phi_k(t, t_k) \circ \Phi_{k-1}(t_k, t_{k-1}) \circ \cdots \circ \Phi_i(t_{i+1}, t_i)$$

$$H_{k,i}(t, t_i) = h_k(t, t_k) + \sum_{j=i}^{k-1} \Psi_{k-1,j}(t_{k-1}, t_j) h_j(t_{j+1}, t_j)$$

where

$$\Psi_{i,i}(t, t_i) = \Phi_i(t, t_i)$$

and

$$H_{i,i}(t, t_i) = h_i(t, t_i)$$

The system evolution for an initial condition $x(T_i)$ can be rewritten as

$$x(t) = \Psi_{k,0}(t, T_i)x(T_i) + H_{k,0}(t, T_i) \tag{7}$$

2.5. *Necessary and sufficient conditions*

We first formally state the result.

Lemma 1: *Under Assumptions* 1–3, *the exact output tracking problem is solvable with bounded solution if and only if the system of equations:*

$$Y_{dk}(t_k) = C_k^* Q_k \Psi_{k-1,0}(t_k, T_i) Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,0}(T_i) \end{bmatrix} + C_k^* Q_k H_{k-1,0}(t_k, T_i), \quad \forall k = 1, \dots, N \tag{8}$$

$$0 = Z_{uN} \left(\Psi_{N,0}(T_f, T_i) Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,0}(T_i) \end{bmatrix} + H_{N,0}(T_f, T_i) \right) \tag{9}$$

admits a solution in $z_{u,0}(T_i)$.

Proof: System trajectories outside $[T_i, T_f]$, the compact support of Y_d, are bounded if and only if z_{uN}, the unstable component of the internal-dynamics, is zero at the end of the motion T_f and similarly the stable component z_{s0} is zero before time T_i. Formally

$$x(T_i) = Q_0^{-1} \begin{bmatrix} 0 \\ 0 \\ z_{u,0}(T_i) \end{bmatrix}$$

$$x(T_f) = Q_N^{-1} \begin{bmatrix} 0 \\ z_{s,N}(T_f) \\ 0 \end{bmatrix}$$

Substitution of the preceding expressions into (7) computed at $t = T_f$

$$x(T_f) = \Psi_{N,0}(T_f, T_i)x(T_i) + H_{N,0}(T_f, T_i)$$

gives (9). In addition, exact tracking in every interval I_k is possible if it is possible to find state trajectories that are continuous and such that $C_k^* x(t_k) = Y_{dk}(t_k)$. By using (7), that gives the state at $t = t_k$ as a function of the initial state and the constraint on $x(T_i)$, (8) easily follows. □

Equation (8) will be referred in the following as compatibility conditions, and (9) will be referred to as stability condition. The compatibility conditions ensure that Y_d does not jump across switches (or else unbounded inputs would be required). The stability condition ensures that the autonomous system dynamics for $t \to \pm\infty$ are bounded.

The algebraic conditions expressed by Lemma 1 can also be interpreted in a geometric coordinate-free framework. To this end, let

$$\mathscr{L}_k = \{x : Y_k = Y_{d,k}(t_k)\}, \quad k \in [0, \ldots, N]$$

represents the set of the admissible system states at time t_k to achieve exact tracking in the time interval $t_k \leqslant t < t_{k+1}$. $\mathscr{L}_k$ is in general a linear variety in the state space that reduces to a linear subspace (the system internal dynamics) for $k = 0$.

A necessary condition for achieving exact tracking when $t < T_i$ is $x_d(T_i) \in \mathscr{L}_0$. Furthermore, to maintain a bounded solution for all $t < T_i$ it is necessary that the initial state belong to the unstable subspace of the system internal dynamics $x_d(T_i) \in \mathscr{L}_{u,0}$.

Note that every $x_d(T_i)$ determines a unique $x_d(t_1)$, given by

$$x_d(t_1) = \Phi_0(t_1, T_i)x_d(T_i) + h_0(t_1, T_i)$$

Hence, we can define the image of the subspace $\mathscr{L}_{u,0}$ as

$$\Phi_0(t_1, t_0) \circ \mathscr{L}_{u,0} = \{x : x = \Phi_0(t_1, t_0)y + h_0(t_1, t_0);\ y \in \mathscr{L}_{u,0}\}$$

which represents the linear variety composed by the points reachable at time t_1 with the constraint of $y(t) = y_d(t)$ for all $t \in [t_0, t_1)$.

To maintain exact tracking in the next interval $t \in [t_1, t_2)$, it is necessary that $x_d(t_1) \in \mathscr{L}_1$. The compatibility condition at time $t = t_1$ states that

$$x_d(t_1) \in \mathscr{L}_1 \cap \Phi_0(t_1, T_i) \circ \mathscr{L}_{u,0}$$

which is possible if and only if the linear variety

$$\mathscr{S}_1 = \mathscr{L}_1 \cap \Phi_0(t_1, T_i) \circ \mathscr{L}_{u.0}$$

is not empty, i.e. if and only if $\mathscr{L}_1$ intersects the image of $\mathscr{L}_{u,0}$ under the system flow. The same procedure can be repeated for the switching time $t = t_2$. Starting from $\mathscr{S}_1$, we can flow forwards in time. To achieve exact tracking in the interval $[t_2, t_3)$, the image of $\mathscr{S}_1$ must intersect $\mathscr{L}_2$, i.e. the set

$$\mathscr{S}_2 = \mathscr{L}_2 \cap \Phi_1(t_2, t_1) \circ \mathscr{S}_1$$

must be not empty and more generally, the exact tracking in $t \in [T_i, t_{k+1})$ is possible if and only if the image of $\mathscr{S}_{k-1}$ under the system flow $\Phi_{k-1}(t_k, t_{k-1})$ intersects $\mathscr{L}_k$, i.e. the set

$$\mathscr{S}_k = \mathscr{L}_k \cap \Phi_{k-1}(t_k, t_{k-1}) \circ \mathscr{S}_{k-1} \tag{10}$$

is non-empty for every $k = 1, \ldots, N$. However, to obtain a bounded solution for $t > T_f$, the final state at time $t = T_f$ must belong to the stable subspace of the system internal dynamics $\mathscr{L}_{s,N}$. Let

$$\mathscr{S}_{T_f} \triangleq \mathscr{L}_{s,N} \cap \Phi_N(T_f, t_N) \circ \mathscr{S}_N$$

Hence, we have proved an analogue of Lemma 1 in geometric terms.

Lemma 2: *The exact output tracking problem is solvable if and only if $\mathscr{S}_{T_f}$ is non-empty, i.e.*

$$\mathscr{S}_{T_f} \neq \varnothing \tag{11}$$

As the flow of a linear system is a homeomorphism, the dimension of a linear variety and that of its image are equal, and hence

$$\dim(\mathscr{S}_i) \geqslant \dim(\mathscr{S}_j), \quad j \geqslant i$$

This means that at each iteration (10) the set of possible solutions could reduce at each K and that no solution is possible if it becomes the empty set for some k, i.e. it is empty for every $j \geqslant k$.

2.6. *Switched systems with invariant internal-dynamics subspace*

We present below a particular case in which the given conditions considerably simplify. This exemplifies the obtained results and will be illustrated with an example in § 5.

Assumption 4: *The system* (1) *has constant relative degree* $r = r_k$ *for every* k, *and matrix* $C_k^* = C^*$ *is constant for every* k.

It follows from the previous assumption that the coordinates outside the internal dynamics Y_k are the same for every k, and thus the internal dynamics subspace is the same for every k. Note that the stable and unstable subspaces may still be different, and may switch around, but are constrained to belong to the same subspace.

As $Y_k = C^*x$, the continuity of x implies that the compatibility conditions are always satisfied.

Lemma 3: *If assumption* 4 *is satisfied, then the compatibility conditions are satisfied for every smooth enough* (C^r) *desired output trajectory* $y_d(t)$.

In addition, $\Phi_k(t_{k+1}, t_k) \circ \mathscr{S}_k \subset \mathscr{L}_{k+1}$ implies that

$$\mathscr{S}_{k+1} = \Phi_k(t_{k+1}, t_k) \circ \mathscr{S}_k$$

is non-empty because $0 \in \mathscr{S}_0$. Further $\dim(\mathscr{S}_n) = \dim(\mathscr{L}_{0.u})$.

The additional condition for boundedness of solutions for $t > T_f$ is met if and only if $\mathscr{S}_N$ intersects $\mathscr{L}_{s,N}$. The linear variety $\mathscr{S}_N$ can be expressed as

$$\mathscr{S}_N = \operatorname{Im}(S_N) + v$$

and the stable subspace of the internal dynamics can be expressed as

$$\mathscr{L}_{s,N} = \operatorname{Im}(L_{s,N})$$

From the previous considerations we have the following lemma.

Lemma 4: *If Assumptions* 1–4 *hold, then the exact output tracking problem with bounded solution is solvable if*

$$\operatorname{rank}\,[S_N \quad L_{N,s}] = n_z \tag{12}$$

where n_z *is the dimension of the system internal dynamics at* $t = T_f$.

As a last remark in this section, note that if not only do the internal dynamics subspace remain constant across the switchings, as ensured by Assumption 4, but its stable and unstable subspaces do too; then the condition (12) is always satisfied and the problem has a solution for every admissible $y_d(t)$. Moreover, this solution is unique for every given $y_d(t)$. This implies that the exact-tracking problem for a system without switches is always solvable.

3. y_d given through an exosystem

In this section we consider how the preceding results can be specified when the reference trajectory is not completely general but is generated through a known linear exosystem, given by

$$\left.\begin{aligned} \dot{x}_e &= A_e(k)x_e \\ y_d &= C_e(k)x_e \end{aligned}\right\} \tag{13}$$

First we solve the tracking problem when the initial state of the exosystem (at time T_i) is known in advance, hence the obtained result will be valid for the particular reference trajectory determined by the initial condition. Later, we will extend the approach to the case of unknown initial conditions. In this case, we look for asymptotic tracking of the reference trajectories for arbitrary initial conditions of the exosystem.

We begin by studying the case where the state of the exosystem $x_e(T_i)$ is known. The system equation (4) becomes

$$\dot{x}_d(t) = A_y(k)x(t) + B_y(k)C_e^*(k)x_e(t) \tag{14}$$

with $y_d^{(r_k)}(t) \triangleq C_e^*(k)x_e(t)$, because all the time derivatives of the output can be written in terms of x_e by using (13).

The system evolution can then be rewritten as

$$x(t) = \Psi_{k,0}(t, T_i)x(T_i) + \hat{H}_{k,0}(t, T_i)x_e(T_i) \tag{15}$$

where

$$\hat{H}_{k,i}(t, t_i) = \hat{h}_k(t, t_k) + \sum_{j=i}^{k-1} \Psi_{k-1,j}(t_{k-1}, t_j)\hat{h}_j(t_{j+1}, t_j)$$

$$\hat{h}_k(t, t_k) = \int_{t_k}^{t} \exp[A_y(k)(t-\tau)]\, B_y(k)C_e^*(k)\hat{\Psi}_{k,0}(\tau, T_i)\, d\tau$$

and $\hat{\Psi}_{k,0}(\tau, T_i)$ is the evolution of the exosystem (analogous to (7)). A solution to the exact tracking problem exists if Lemma 2 is satisfied. The compatibility condition is satisfied for all initial conditions $x_d(T_i) \in \mathscr{L}_0$ if $C_e^* = C^*$. The stability condition becomes

$$0 = s_1 z_{u,0}(T_i) + s_2 x_e(T_i) \tag{16}$$

where

$$s_1 = Z_{uN}\Psi_{N,0}(T_f, T_i)Q_0^{-1}\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

$$s_2 = Z_{uN}\hat{H}_{N,0}(T_f, T_i)$$

In what is to follow, we will assume that the above equation has a unique solution (iff s_1 is invertible). This yields a one-to-one relationship between the plant's state and the exosystem, as follows:

$$x_{\rm d}(T_i) = Q_0^{-1}\begin{bmatrix} 0 \\ 0 \\ z_{u,T_i}(T_i) \end{bmatrix}$$

$$= Q_0^{-1}\begin{bmatrix} 0 \\ 0 \\ -s_1^{-1}s_2x_{\rm e}(T_i) \end{bmatrix} \triangleq G(T_i)x_{\rm e}(T_i)$$

What is interesting is that we can also write the desired exact tracking state trajectory in terms of the exosystem state. By substituting the above expression into (15) we obtain

$$x_{\rm d}(t) = G(t)x_{\rm e}(t)$$

where

$$G(t) \triangleq [\Psi_{k,T_i}(t, T_i)G_{(T_i)} + \hat{H}_{k,T_i}(t, T_i)C_{\rm e}^*]\hat{\Psi}_{k,0}^{-1}(N, T_i)x_{\rm e}(t) \tag{17}$$

It may be verified that $G(t)$ satisfies the differential equation

$$\dot{G}_k(t) = A_y(k)G_k(t) - G_k^{\rm T}(t)A_{\rm e}(k) + C_{\rm e}^*$$

In the case of no switching, a constant solution always exists for the above Lyapunov equation provided that the eigenvalues of the exosystem $A_{\rm e}$ are different from the zeros of the plant eigenvalues of A_y. The above equation also provides a control strategy when the exosystem states are not known. We estimate the state of the exosystem as $\hat{x}_{\rm e}$ and regulate the trajectory $\hat{x}_{\rm d} = G(t)\hat{x}_{\rm e}$. The stability of such a controller is studied in the next section.

4. Stabilization

If the state of the exosystem $x_{\rm e}$ is not known, then we could estimate it as $\hat{x}_{\rm e}$ with $\|x_{\rm e}(t) - \hat{x}_{\rm e}(t)\|_2 \leqslant K_{\rm e}\, {\rm e}^{\alpha_{\rm e}t}\|x_{\rm e}(0) - \hat{x}_{\rm e}(0)\|_2$. We use $\hat{x}_{\rm d}(t) = G(t)\hat{x}_{\rm e}(t)$ as the estimated desired state trajectory, and stabilize this trajectory by using the control scheme shown in Fig. 3. Note that the feedforward used (see (3)) is completely specified in terms of the exosystem's state estimate, as follows:

$$\begin{aligned} u_{\rm d} &= F_{k,1}x_{\rm d}(t) + F_{k,2}y_{\rm d}^{(r_k)}(t) \\ &= F_{k,1}G(t)\hat{x}_{\rm e}(t) + F_{k,2}C_{\rm e}^*\hat{x}_{\rm e}(t) \\ &\triangleq F_k\hat{x}_{\rm e}(t) \end{aligned}$$

The state equations are of the form

$$\dot{x} = A_kx + B_k(F_k\hat{x}_{\rm e} + K(x - G(t)\hat{x}_{\rm e}))$$

We require that the system in each interval is either stable or stabilizable. For simplicity, we assume that $A(k) + B(k)K$ is Hurwitz for all k; the arguments remain valid if the system is stabilized through any other feedback control scheme.

The desired trajectory satisfies

$$\dot{x}_{\rm d} = A_kx_{\rm d} + B_kF_kx_{\rm e}$$

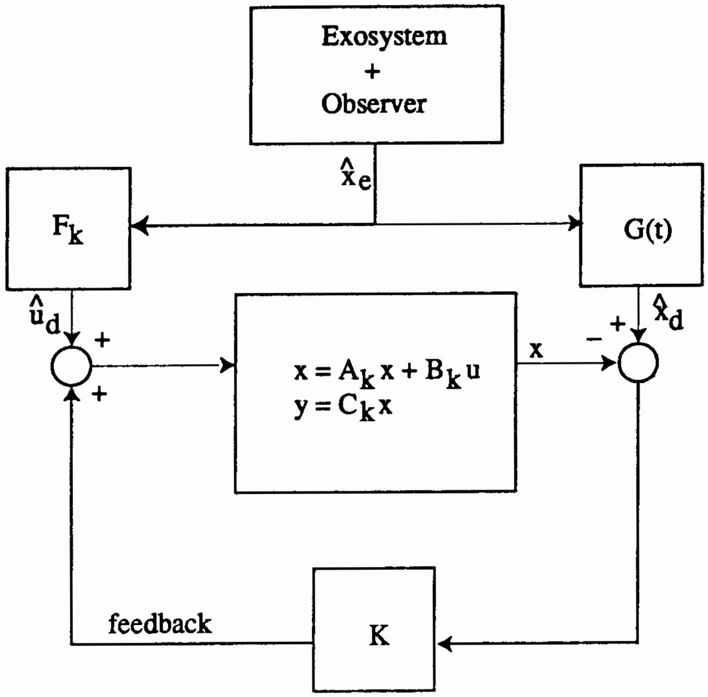

Figure 3. Trajectory tracking with exosystem.

Let $e := x - x_d$. Then the difference between the last two equations yields

$$\dot{e} = (A_k + B_k K)e + (B_k F_k + B_k K G(t))(\hat{x}_e - x_e)$$

The exponential stability of the error dynamics system follows from the next lemma.

Lemma 5: *Given the system $\dot{e} = Ae + v(t)$, where $K_1 e^{\alpha_1 t} < \|e^{-At}\|_2 < K_2 e^{\alpha_2 t}$ and $\|v(t)\|_2 < K_3 e^{-\alpha_3 t}$, with K_1, K_2, K_3 positive, then $e = 0$ is an exponentially stable equilibrium point, provided that $\alpha_3 > \alpha_2 > 0$.*

Proof: Using the variation of constants formula

$$\begin{aligned}
\|e^{-At} e(t)\|_2 &\leqslant \left\| e(0) + \int_0^t e^{-A\tau} v(\tau)\, d\tau \right\|_2 \\
&\leqslant \|e(0)\|_2 + \left\| \int_0^t e^{-A\tau} v(\tau)\, d\tau \right\|_2 \\
&\leqslant \|e(0)\|_2 + \int_0^t \|e^{-A\tau} v(\tau)\|_2\, d\tau \\
&\leqslant \|e(0)\|_2 + K_2 K_3 \int_0^t e^{\alpha_2 - \alpha_3}\, d\tau \\
&\leqslant \|e(0)\|_2 + K_2 K_3 \frac{1}{\alpha_3 - \alpha_2}
\end{aligned}$$

provided that $\alpha_3 > \alpha_2$. Hence

$$K_1 \|e^{\alpha_1 t} e(t)\|_2 \leqslant \|e^{-At} e(t)\|_2 \leqslant \|e(0)\|_2 + K_2 K_3 \frac{1}{\alpha_3 - \alpha_2}$$

the right-hand side being a constant. Therefore, for all $\epsilon > 0$ there exists a positive constant K such that

$$\|e(t)\|_2 < K\,\mathrm{e}^{-(\alpha_1-\epsilon)}$$

5. Example

Consider the flexible structure, cantilevered at the base and free at the top, shown in Fig. 4. It is modelled (with the finite-element method) with a single flexural element. The degrees of freedom are the translational motion at the base x_1, and the top x_2, and the rotation at the top x_3. The input is a translational force at the base and the system output is x_2. The structure is loaded with a mass m_t, which is changed at several instances. The flexural element has the following properties: mass 420; length 1; elastic modulus 1; and cross-sectional area moment of inertia 1. The objective is to maintain the top of the structure along a prescribed trajectory to facilitate the transfer of the load. The equations of motion can be described by

$$M\ddot{x} + Sx = \hat{B}u$$

where

$$M = \begin{bmatrix} 156 & 54 & -13 \\ 54 & 156+m_t & -22 \\ -13 & -22 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 12 & -12 & 6 \\ -12 & 12 & -6 \\ 6 & -6 & 4 \end{bmatrix}$$

$x = [x_1\ x_2\ x_3]'$, and $\hat{B} = [1\ 0\ 0]'$.

In the standard form $x = [x'\ \dot{x}']'$ (abuse of notation) we have the dynamics as

$$\dot{x} = A_k x + B_k u$$
$$y_k = C_k x$$

where

$$A_k = \begin{bmatrix} 0 & I \\ -M^{-1}S & 0 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0 \\ M^{-1}\hat{B} \end{bmatrix}, \quad C_k = [0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$$

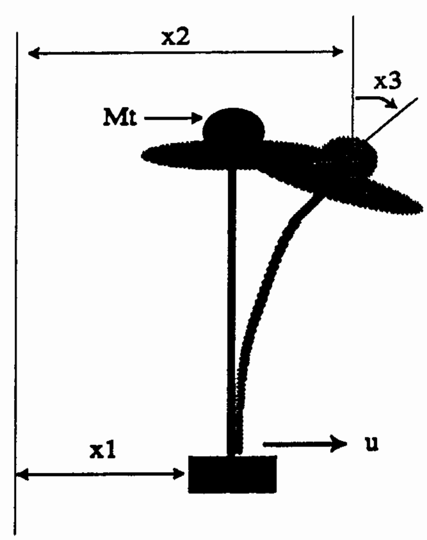

Figure 4. Example.

The desired trajectory is generated by an exosystem of the form $\dot{x}_e = A_e x_e$, where

$$A_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and the desired output is $y_d = [0\ \ 0\ \ 0\ \ 1]x_e \triangleq C_e x_e$. We also switch the exosystem to $A_e = 0^{4\times 4}$ at the initial and final times $T_i = 0$ and $T_f = 2pi$. The first two states of the exosystem form an oscillator and the second state is then integrated twice to obtain the desired output. Note that (A_e, C_e) is observable. Hence the exosystem states can be estimated. In our simulations we ensure that the output trajectories have a compact support $[0, 2\pi]$ by choosing initial conditions of the exosystem of the form $x_e(T_i) = [0; *; 0; 0]$.

We also switch the mass m_t on the structure (see Fig. 4) to take the values $m_t = 0$, $\forall t \in [0, \pi/2]$, $m_t = 100$, $\forall t \in (\pi/2, 1{\cdot}5\pi]$, and $m_t = 10$, $\forall t \in (1{\cdot}5\pi, 2\pi]$, which denotes the jumps in the system. Note that C_k^* remains constant through the switches and hence the compatibility conditions are always satisfied. As illustrated in § 3, the map $G(0) : x_e(T_i) \rightarrow x_d(T_i)$ is given as

$$G(0) = \begin{bmatrix} -4{\cdot}6715 & -3{\cdot}3239 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 6{\cdot}9406 & 2{\cdot}6672 & 0 & 0 \\ -0{\cdot}5935 & -1{\cdot}4710 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2{\cdot}6851 & -0{\cdot}8211 & 0 & 0 \end{bmatrix}$$

As an example we simulate the forward dynamics with the initial condition for the exosystem as $[0\ \ 1\ \ 0\ \ 0]'$. The corresponding initial system state for exact output tracking is

$$x_d(T_i) = [-3{\cdot}3239 \quad 0 \quad 2{\cdot}6672 \quad -1{\cdot}471 \quad 0 \quad -0{\cdot}8211]'$$

The simulation results are shown in Fig. 5, where the exact tracking state trajectory is shown; this desired state trajectory yields the desired output with an error of 10^{-6} for a motion of two units. This error is believed to be due to the numerical integration schemes. Furthermore, the initial conditions are large and unrealistic. The initial conditions of the system are typically not the same as the initial conditions of the desired state trajectory; this results in initial transient errors. If the system is stabilized then these errors decay exponentially, even if the system dynamics has switches. This ability to maintain tracking across switches is a major advantage of our approach. Furthermore, preactuation techniques to achieve these initial conditions (with output error maintained at zero) has been developed by Devasia *et al.* (1996) and we expect to integrate the two approaches in the future.

6. Conclusions

The problem of achieving exact output tracking for linear systems that present jumps in a parameter's values has been analysed. We have established necessary and sufficient conditions for the existence of exact output-tracking bounded state trajectories. When the reference trajectory is generated through an exosystem, the

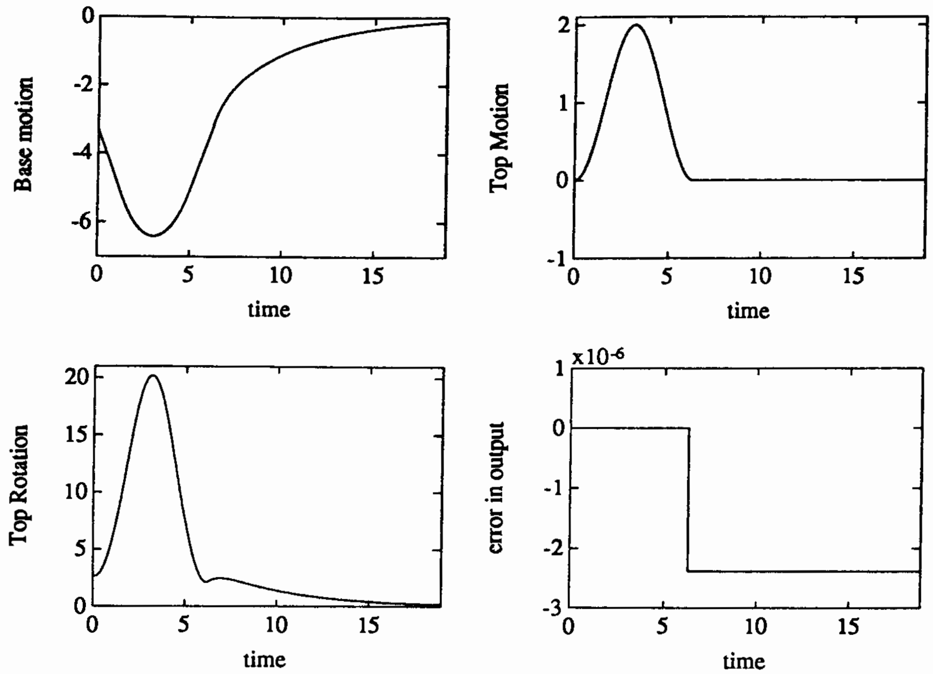

Figure 5. Simulation results.

feedforward action needed to maintain exact tracking can be written as a time-varying feedback. Furthermore, this time-varying feedback is related to a map from the state of the exosystem to the desired system state. The map is linear and is shown to satisfy an ordinary differential equation. For the case of systems without switches the presented theory reduces to the standard regulator theory. We also showed that the desired trajectory can be stabilized and presented the simulation results for an example flexible structure with switching mass.

Future work will attempt to remove the requirement of compact support for the output. There is also a need to address the tracking problem for systems whose internal dynamics may not be hyperbolic.

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