State Observer for Linear Time-Invariant Systems With Quantized Output

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In this paper we introduce a state observer for linear time-invariant systems with quantized outputs. The observer employs an orthogonal projection operation at quantizer output discontinuities to enhance its convergence rate for stable systems. The increasing rate of convergence and stability has been proven by using Lyapunov second method. Some sufficient and necessary conditions of stability for the unstable systems are derived. The sufficient condition of noise stability is given and the maximal bound of noise stability is presented. The proposed methodology has been applied to state estimation of a DC-motor with optical encoder.

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Contributed by the Dynamic Systems and Control Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received by the Dynamic Systems and Control Division December 18, 1995. Associate Technical Editor: S. D. Fassois.

1 Introduction

In all digitally controlled plants, measured outputs are quantized prior to control computation. In most cases, the quantization error is small compared to system noise and is justifiably ignored. There are exceptions, however. One notable example is motor control where an optical encoder provides the only measured output, and mechanical position noise of the shaft due to vibration etc. is small as compared to quantization errors. In this paper we address the observer design problem for such linear time-invariant systems with quantized outputs. We show that incorporation of knowledge of the quantization nonlinearity leads to an improvement in the state estimate with a minor increase in observer complexity. Quantization has been addressed in the control systems context by several researchers. Curry (1970) has developed maximum likehood estimates for static linear systems driven by Gaussian noise and having quantized outputs. The extension to linear dynamic systems appears intractable analytically, however, Curry does derive approximate formulae for state estimates that work well with small quantizer steps. Another approach proposed by Schewppe (1968) propagates an ellipsoidal set which approximates the true system state by containment. This method only requires knowledge of bounds on inputs, and bounds on output measurement (quantizer) error; the performance calculation is intractable analytically in this case also. More recently, Miller et al. (1989) establish useful bounds on tracking performance in digitally controlled plants, where there is numerical quantization in the digital computation of the control input. Delchamp (1988) takes a new and fundamental approach to dealing with quantization. Rather than treating quantization as a bounded disturbance, his method treats quantization exactly, in the linear dynamic case, and establishes conditions under which the uncertainty in the system state tends to zero (as measured by differential entropy). The approach uses the system input to optimize information acquired on the state, but mixed approaches aimed at addressing information and tracking performance simultaneously now appear as possibilities. More recent works by Rotea and Williamson (1994) and others are representative of a broad class of problems focused on choosing state-space realization of discrete-time linear time-invariant systems which perform well in computer implementations. These methods effectively treat numerical round-off quantization as a noise source and address the scaling of internal signals to optimize competing objectives of (*a*) low sensitivity to quantization and (*b*) the desire for infrequent numerical overflow. The approach contrasts with the authors where quantization is modeled as a nonlinearity, rather than a noise source. Moreover, we necessary work with continuous systems rather than discrete-time systems. More recent work addressing chaos in feedback systems with quantization is due to Steppan and Haller (1996).

This paper has the following format. In Section 2 we motivate and introduce the observer of SISO and MIMO systems. In Section 3 we modify existing Lyapunov theory to accommodate the discontinuous updates used in our observer, and prove error convergence for a stable plant. Some sufficient and necessary conditions of stability for an unstable system are given. The maximal bound of noise stability is presented. Section 4 contains simulation results for the stable and unstable DC-motor with optical encoder. Our conclusions are made in Section 5.

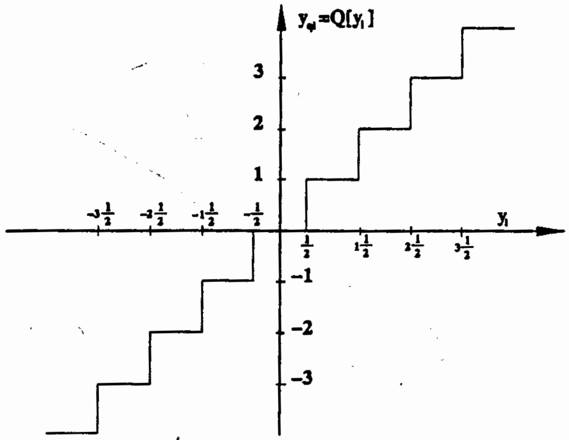

Fig. 1 Quantizer nonlinearity

2 Projection-Based Observer

A MIMO linear time-invariant system with quantized outputs is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$
$$y(t) = Cx(t)$$
$$y_q(t) = Q[y(t)], \quad \forall t \in \{t_k\}_{k=0}^{\infty} \tag{1}$$

where $x \in R^n$, $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{n \times q}$ and t_k is quantizer transition time, $y \in R^q$ is continuous output, and $y_q \in R^q$ is quantized output. The function $Q[\cdot]$ is the component-wise modified quantizer nonlinearity as shown in Fig. 1.

2.1 Single Output Case. First we will consider the single-output quantized systems, $y \in R$. In our observer design, the discontinuities play a critical role since the only times that y is measured exactly is at discontinuities. At other times, it differs from y by up to 1 as shown in Fig. 1. To see how the quantizer discontinuities can be used to estimate the state $\hat{x}$, consider Fig. 2. In this figure the state space is represented as the direct sum of range of C and the nullspace of C^T. A full rank matrix Γ is chosen such that $R(\Gamma) = N(C^T)$. In addition, let M satisfy the Lyapunov equation $A^T M + MA = -I$, and define a natural inner product on the state space by $\langle x, y \rangle_M \equiv x^T M y$. In this Hilbert space, the orthogonal projection matrix onto $N(C^T)$ is given by $P = \Gamma(\Gamma^T M \Gamma)^{-1} \Gamma^T M$. The corresponding projection onto $N(C^T)^{\perp M} = R(M^{-1}C)$ is $I - P$. At a quantizer transition, the value of y is known exactly to be, say, y_q. As a consequence, the state satisfies the equation $y_q = Cx$ and lies on the hyperplane, as shown in Fig. 2. If $\hat{x}$ is the present estimate of the state, the estimate can be improved by projecting it along $R(M^{-1}C)$ to the nearest point (with respect to $\langle \cdot, \cdot \rangle_M$) in the hyperplane containing x. This is the basic projection step used with our observer. The projection can be calculated in terms of $y_q = Q[y]$ to be

$$\hat{x}_{\text{new}} = \hat{x} - M^{-1}C(C^T M^{-1} C)^{-1}(\hat{y} - y_q) \tag{2}$$

Let t_k be the time at which a quantizer transition occurs, and let y_q be the corresponding transition value. We define our observer by

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \forall\, t \notin \{t_k\}_{k=0}^{\infty}$$
$$\hat{x}_{\text{new}} \leftarrow \hat{x} - M^{-1}C(C^T M^{-1} C)^{-1}(\hat{y} - y_q), \quad \forall\, t\{t_k\}_{k=0}^{\infty} \tag{3}$$

where the arrow "$\leftarrow$" indicates a discrete update. We are able to prove error convergence by using modification of nonlinear Lyapunov theory of Khalil (1992) in the simple case when A is Hurwitz. When A is unstable we derived some sufficient and necessary conditions for Lyapunov stability analysis. A formal proof of convergence for the general case has been found in terms of quantization time intervals and eigenvalues of the system.

2.2 Multi-Output Case. Herein we will extend our methodology of the design state observer for MIMO linear time-invariant systems with quantized outputs. The equations of the observer based on the projection algorithm can be expressed by

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \forall\, t \notin \{t_k\}_{k=0}^{\infty}$$
$$\hat{x}_{\text{new}} \leftarrow \hat{x} - M^{-1}C_i(C_i^T M^{-1} C_i)^{-1}(\hat{y}_i - y_{qi})$$
$$\forall t\{t_{ki}\}_{k=0}^{\infty}, \quad \forall i = 1, 2 \ldots, q \tag{4}$$

where $C \in R^{n \times q}$, $i = 1, 2, \ldots, q$ is output numbers, and $A^T M + MA = -I$. The case of multi-output is increasing the information of outputs and the number of projections. Suppose all quantized outputs are measured at quantizer transition simultaneously, then all outputs can be applied to Eq. (4) in order to compute $\hat{x}_{\text{new}}$ simultaneously. However, all quantized outputs do not have quantizer transition synchronously in general. The case of multi-output is increasing the information of outputs and the number of projections. Intuitively, the estimation error of state observer based on the multiple outputs should have an improved convergence rates relative to a single output system. In Eq. (4), t_{ki} is the time of the kth transition on the ith output. When multiple quantizer transitions occur at the same time, the respective projections are executed in arbitrary order. More sophisticated schemes can be derived for simultaneous transitions, but for practical families of state trajecteries with associated probability measures, the probability of simultaneous transitions is zero.

3 Stability Analysis and Noise Rejection

In this section, we formulate the stability problem and give a proof of convergence for the SISO case when A is stable or unstable (the MIMO case follows directly). We first define a solution to a differential equation with resetting at time t_k. Consider the differential equation with resetting

$$\dot{x} = f(x, t)$$
$$x(t_k) \leftarrow x_k, \quad \forall k = 0, 1, \ldots, n \tag{5}$$

where $n \in \{R, \infty\}$, and there are finite t_k in any finite interval. Define be the caratheodory solution to $\dot{x} = f(x, t)$, $x(t_k) = x_k$ on the interval $[t_k, t_{k+1})$ and so $x(t)$ is defined on all of $[t_0, \infty]$ provided $f(x, t)$ satisfies the assumption of piecewise continuity in t and local Lipschitz condition. Define $x(t_k^-) = \lim_{t \to t_k} x(t)$. Lyapunov stability theory for systems of the form (5) is easily generalized from standard results. All that is required to deal with the resetting is to require that a Lyapunov function not increase upon a reset. As an illustration, we generalize theorem 4.1 of Khalil's book (1992). Let $f(x, t)$ be Lipschitz continuous on a domain $D = \{x \in R \mid \|x\| < r\}$, then we have the following.

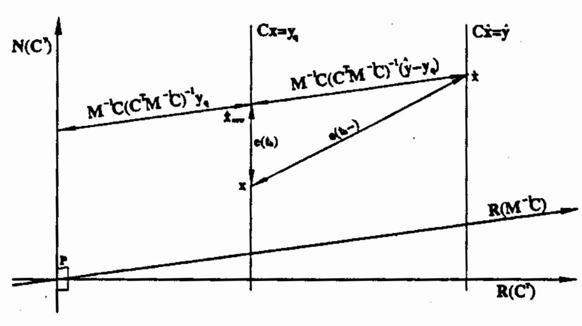

Fig. 2 The projection step in the observer

Theorem 1. Consider system (5) where t_k is a finite or countable set of resetting times. Let $x = 0$ be an equilibrium point of $\dot{x} = f(x, t)$, and $D = \{x \in R \mid \|x\| < r\}$. Let V: $[0, \infty) \times D \rightarrow R$ be a continuously differential function such that

$$\alpha_1(\|V(t, x)\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{6}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \leq -\alpha_3(\|x\|), \quad \forall\, t \notin \{t_k\} \tag{7}$$

$$V(x(t_k, t_k) - V(x(t_{k^-}), t_k) \leq 0, \quad \forall t \in \{t_k\}, \quad \forall x \in D \tag{8}$$

where α_1, α_2, and α_3 are class K function defined on $[0, r)$. Then $x = 0$ is uniformly asymptotically stable.

Proof: See the reference of Khalil (1992). The only change in the proof required is that (8) in addition (7) is required to show that V is decreasing.

Corollary 1. If all of the assumptions of the theorem are satisfied with $\alpha_i(r) = k_i r^c$, for some positive constants k and c, then $x = 0$ is exponentially stable. Moreover, if the assumptions hold globally, then $x = 0$ is globally exponentially stable.

Note that we have not formally defined the various kinds of stability for the differential equation with resetting, but the generalizations are so slight that this is not required. Return now to the combined SISO plant observer system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$y_q = Q[y] \tag{9}$$

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \forall t \notin \{t_k\}_{k=0}^{\infty}$$

$$\hat{y} = C\hat{x}$$

$$\hat{x}_{\text{update}} \leftarrow \hat{x} - M^{-1}C(C^TM^{-1}C)^{-1}(\hat{y} - y_q), \quad \forall t \in \{t_k\}_{k=0}^{\infty} \tag{10}$$

Define the state error $e = x - \hat{x}$ and Lyapunov function $V(e) = e^TMe$. Then for $t \notin \{t_k\}$ we have $\dot{V}(e) = -e^Te = -\|e\|^2$. For $t \in \{t_k\}$ we have $V(e(t^+)) - V(e(t)) = -(\hat{y} - y_q)^T (C^TM^{-1}C)^{-1}(\hat{y} - y_q) \leq 0$. It follows that $e \rightarrow 0$ esponentially. The rate of convergence is hard to define since we cannot quantify the rate at which quantizer transitions occur. Simulations of a DC-motor with optical encoder example show that the projection operation increases the rate of convergence relative to Eqs. (9) and (10). In the stable system, the Lyapunov function is monotonically decreasing along the solutions of the system and "forced down" by discrete updates to converges to zero more quickly as t tends to infinity. In case that the monotone property of Lyapunov function along solutions of the error system is no longer holds for unstable systems. We need another condition for stability given in the following theorem.

Theorem 2. The observer based on the projection algorithm (10) is exponentially stable if there exist $P \in R^{n \times n}$, $\lambda > 0$ and $\Delta T > 0$ such that

$$e^{A^T\Delta t_k}(I - P)^TM(I - P)e^{A\Delta t_k} - e^{-\lambda\Delta t_k}M < 0, \quad \forall \Delta t_k \leq \Delta T$$

where $P = M^{-1}C(C^TM^{-1}C)^{-1}C^T \in R^{n \times n}$ and Δt_k are finite quantized time intervals.

Proof: Let $e = x - \hat{x}$ be state error, then we have the error equation of (9) and (10) as following.

$$\dot{e} = Ae, \; e(t_0) = e_0, \quad \forall t \notin \{t_k\}_{k=0}^{\infty}$$

$$e_{\text{new}}(t) \leftarrow (I - P)e(t), \quad \forall t \in \{t_k\}_{k=0}^{\infty} \tag{11}$$

And the solution is

$$e(t) = e^{A\alpha} \prod_{i=1}^{k} (I - P)e^{A\Delta t_k}e_0 \tag{12}$$

where

$$t = a + \sum_{i=1}^{k} \Delta t_i, \quad 0 \leq \alpha < \Delta t_{k+1} \quad \text{and} \quad \Delta t_k = t_{k+1} - t_k.$$

Let $V(k, e) = e(k)^TMe(t)$ be Lyapunov function of error equation. Then the Lyapunov function at $k + 1$ can be expressed by

$$\begin{aligned} V(k + 1, e) &= e(k + 1)^TMe(k + 1) \\ &= e(k)^Te^{A^T\Delta t_k}(I - P)^TM(I - P)e^{A\Delta t_k}e(k) \end{aligned} \tag{13}$$

Suppose that $e^{A^T\Delta t_k}(I - P)^TM(I - P)e^{A\Delta t_k} - e^{-\lambda\Delta t_k}M < 0$. Then we have

$$e^{A^T\Delta t_k}(I - P)^TM(I - P)e^{A\Delta t_k} < e^{-\lambda\Delta t_k}M$$

$$e(k)^Te^{A^T\Delta t_k}(I - P)^TM(I - P)e^{A\Delta t_k}e(k) < e^{-\lambda\Delta t_k}e(k)^TMe(k)$$

$$V(k + 1, e) < e^{-\lambda\Delta t_k}V(k, e)$$

$$\frac{V(k + 1, e}{V(k, e)} < e^{-\lambda\Delta t_k}, \quad \forall k \in Z \tag{14}$$

Thus $\lim_{k\rightarrow\infty} V(k, e)$ converge to zero exponentially. Therefore, $e(t)$ converges to zero exponentially as $t \rightarrow \infty$.

In many physical systems, noise is introduced in the measurement device. We consider the following quantized output systems with a measurement noise

$$\dot{x} = Ax + Bu$$

$$y_n = Cx + n(t)$$

$$y_{qn} = Q[y] \tag{15}$$

where $n(t)$ is output measurement noise, $\|n(t)\| \leq N$, for all time. In order to reject the output measurement noise in this methodology, we use a partial orthogonal projection as follows

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \forall t \notin \{t_k\}_{k=0}^{\infty}$$

$$\hat{x}_{\text{new}} \leftarrow \hat{x} - \beta M^{-1}C(C^TM^{-1}C)^{-1}(\hat{y} - y_{qn}), \quad \forall t\{t_k\}_{k=0}^{\infty} \tag{16}$$

where $\beta(\beta \leq 1)$ is a positive scalar gain of projection, and $y_{qn}(t) = Q[y_n(t)] = Q[y(t) + n(t)]$ output plus bounded noise at time t_k. The following theorem shows that the estimation error can be bounded for the system (15) with bounded measurement noise $\|n(t)\| \leq N$.

Theorem 3. Suppose that measurement noise $n(t)$ is bounded. Then there exist a positive constant E such that $\|e(t)\| \leq E$.

Proof: Let $e = x - \hat{x}$ be state error, then we have Lyapunov function $V(e) = e^TMe$. Since system (15) is stable, we have $\dot{V}(e) = -e^Te = -\|e\|$. For $t \in \{t_k\}$ we have

$$\begin{aligned} &V(e(t^+)) - V(e(t)) \\ &= -e^T(t^+)Me(t^+) - e^T(t)Me(t) \\ &= -e^T[M - 2\beta C(C^TM^{-1}C)^{-T}C^T \\ &\quad + \beta^2C(C^TM^{-1}C)^{-1}C^T]e \\ &\quad - 2e^T[\beta C(C^TM^{-1}C)^{-1}(I - \beta I)]n \\ &\quad + \beta^2n^T(C^TM^{-1}C)^{-1}n - e^TMe \\ &\quad - e^T[\beta C(C^TM^{-1}C)^{-1}(I - \beta I)C^T]e \\ &\quad - 2e^T[\beta C(C^TM^{-1}C)^{-1}(I - \beta I)]n \\ &\quad + \beta^2n^T(C^TM^{-1}C)^{-1}n \end{aligned}$$

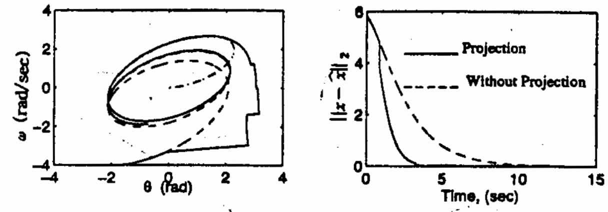

Fig. 3 The state trajectories and estimated error norm of a DC-motor with and without projection operation

$$= -(y - \hat{y})^T[\beta(C^TM^{-1}C)^{-1}(I - \beta I)](y - \hat{y}) - 2(y - \hat{y})^T[\beta(C^TM^{-1}C)^{-1}(I - \beta I)]^n + \beta^2 n^T(C^TM^{-1}C)^{-1}n \quad (17)$$

since $e(t^+) = (I - \beta M^{-1}C(C^TM^{-1}C)^{-T}C^T)e(t) - M^{-1}C(C^TM^{-1}C)^{-T}\beta n(t)$. From Eq. (17), $V(e(t^+)) - V(e(t))$ is negative for large $e(t)$, and bounded measurement noise ($n(t) \leq N$). Therefore there exist a positive constant E such that $\|e(t)\| \leq E$. The estimation error of measurement noise can be minimized as choosing a proper gain β. Herein we did not prove yet.

4 Application to Motor Control

To assess the value of the projection operation in our observer we simulate the simple *DC*-motor with optical encoder modeled by

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix} x & 0\\ 1 & -\dfrac{\alpha}{\beta}\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} + \begin{bmatrix}\gamma\\ 0\end{bmatrix}u$$

$$y_q = Q[\theta] \quad (18)$$

where θ is angular position, ω is angular velocity, and x, α, β are known coefficients of the *DC*-motor in Friendland's book (1986). The quantized output from the optical encoder is $y_q(t_k) = Q[Cx(t_k)]$, and is used to calculate $\hat{x}_{\text{new}}(t_k)$ in order to reset for designed motor observer using results of Section 2. The simulation results of the *DC*-motor system with optical encoder are shown in Fig. 3. When the estimated error of the closed-loop system using the projection operation decays to zero more quickly than without projection. The projection approach for designing an observer is easy and profitable for systems with quantized outputs. One of the difficulties in the design of an observer based on the projection algorithm for unstable system is to find symmetric positive definite matrix $M = M^T > 0$ which satisfies the stability conditions for given transition time intervals as shown in the previous section.

5 Conclusion

The objective of this paper has been to demonstrate that a quantized measurement may be viewed profitably as limited information to extract the estimated state using the projection algorithm. The quantized output measurement can be taken into account explicitly for observer design. Choosing a proper projection operation in Hilbert space, we have shown that the state estimation error can reduced with update logic for the estimated state using an orthogonal projection operation (also in Sur's paper (1996)). As shown Theorems 1 and 2, the observer for the quantized output system has the advantage that the estimated error goes to the zero as $t \to \infty$ inspite of limited output information. Also, we have developed the theory of stability for observer with output noise using the Lyapunov method. Our methodology has been applied to observer design of a *DC*-motor with optical encoder. The results of numerical analysis of *DC*-motor system seem excellent and show that we can update the control law more efficiently to meet various control objectives inspite of limited output information. Finally, we suggest that it is valuable to develop an observer for nonlinear systems with quantized outputs.

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