a fixed order. Furthermore, the label-correcting methods developed are parallelizable as in [6] and will likely lead to efficient parallel algorithms. Finally, the label-correcting methods we presented may also be used in the case where the cost function is of the form $r(x, u)$. In this case the theory of [10] cannot be applied and a Dijkstra-like algorithm is not possible. However, the label-correcting algorithms we proposed can be used as heuristics that specify the order in which the label updates are performed. Their efficient implementation is an interesting subject for further research.

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Stable Inversion for Nonlinear Nonminimum-Phase Time-Varying Systems

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***Abstract*— In this paper, we extend stable inversion to nonlinear time-varying systems and study computational issues—the technique is applicable to minimum-phase as well as nonminimum-phase systems. The inversion technique is new, even in the linear time-varying case, and relies on partitioning (the dichotomic split of) the linearized system dynamics into time-varying, stable, and unstable, submanifolds. This dichotomic split is used to build time-varying filters which are, in turn, the basis of a contraction used to find a bounded inverse input-state trajectory. Finding the inverse input-state trajectory allows the development of exact-output tracking controllers. The method is local to the time-varying trajectory and requires that the internal dynamics vary slowly; however, the method represents a significant advance relative to presently available tracking controllers. Present techniques are restricted to time-invariant nonlinear systems and, in the general case, track only asymptotically.**

***Index Terms*—Dynamics, feedforward systems, inverse problems, tracking.**

I. INTRODUCTION

In this paper, the stable inversion problem for nonlinear time-varying systems is solved. The approach is quite novel in that it applies to nonminimum phase systems—even the linear version of our approach is new in the time-varying context. The basic idea is to compute the inverse dynamics, through a contraction, to find an input-state trajectory that achieves a desired output trajectory. To develop output tracking controllers, the input trajectory (found through inversion) can be used as a feedforward input signal in conjunction with any conventional feedback control law that stabilizes the inverse state trajectory [1]. The present work completes a line of research which was motivated by the inversion of time-invariant articulated flexible structure dynamics [2] and extends our work on inversion of general affine-in-control time-invariant nonlinear systems [3].

System inversion is key to recent results in exact-output tracking for autonomous systems [1], [3]–[5]. This paper extends these results to exact-output tracking of time-varying systems. The output tracking problem has a long history marked by the solution of the linear time-invariant regulator problem by Francis [6] and the nonlinear time-invariant generalization made by Isidori and Byrnes [7]. The linear regulator is designed by solving a manageable set of linear matrix equations, whereas the nonlinear regulator requires the nontrivial solution of a first-order partial differential algebraic equation. These approaches asymptotically track any member in a given family of output signals. More recently, there have been refinements of these approaches. Huang and Rugh [8] used a formal Taylor series expansion of the Isidori–Byrnes partial differential equation and gave a sufficient condition for solvability. Krener [9] extended this work by providing necessary and sufficient conditions for the term-by-term

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solvability of the Taylor series. Robustness issues are studied by Huang and Rugh [10] and Huang and Lin [11]. Other methods that result in approximate tracking can be found in [12]–[14].

The Francis and Isidori–Byrnes regulators apply to time-invariant systems and have the property that they track any one of a family of signals asymptotically. Our approach trades the requirement of solving the partial differential algebraic equation encountered in the Isidori–Byrnes regulator with the requirement of tracking a specific trajectory (rather than any one of a family). Moreover, no exosystem is required, and the specification of trajectories is simplified. We do, however, introduce boundedness and integrability requirements on the output trajectory. The key to our approach is finding a bounded inverse, even for nonminimum phase nonlinear systems, for use in generating feedforward inputs. Since inversion is key to our method, the works of Hirschorn [15] and Di Benedetto and Lucibello [16], [17] on inversion are most relevant to this paper. The early work of Hirschorn is restricted to causal inverses of time-invariant systems and agrees with our inverse in this restricted case. Di Benedetto and Lucibello consider the (nonlinear time-varying) system's initial condition as an input and use inversion in this context; the difference is that we, in effect, use noncausal input to set up the desired initial condition and provide the framework for constructing the noncausal input.

Noncausal feedforward can be used if trajectory preview is possible or truncated to a causal signal at the cost of introducing transient tracking errors [3]. Such noncausal character is seen in the linear quadratic setting, but the use of exact inverses in nonlinear tracking control is new. The noncausal inverses used here have been applied to the control of flexible-link robots in [18]. A recent work by Meyer *et al.* [4] makes extensions in the context of air-traffic control.

More concretely, consider a nonlinear system described in the following normal form [19]:

$$\begin{aligned} y^{(r)}(t) &= \alpha[\zeta(t), \eta(t), t] + \beta[\zeta(t), \eta(t), t]u(t) \\ \dot{\eta}(t) &= s_1[\zeta(t), \eta(t), t] + s_2[\zeta(t), \eta(t), t]u(t) \end{aligned} \tag{1}$$

where $t \in \Re^1$ represents time, $r = (r_1, r_2, \cdots, r_p)^t$ is a vector of positive integers, $y(t) = [y_1(t), y_2(t), \cdots, y_p(t)]^T$ is the output, ζ represents the output y, along with the output's time derivatives, and

$$y^{(r)}(t) \triangleq \left[\frac{d^{r_1}y_1}{dt^{r_1}}(t), \frac{d^{r_2}y_2}{dt^{r_2}}(t), \cdots, \frac{d^{r_p}y_p}{dt^{r_p}}(t)\right]^T.$$

Further, $\alpha(\cdot, \cdot, \cdot)$ and $\beta(\cdot, \cdot, \cdot)$ are smooth in their arguments with $\alpha(0, 0, t) = 0$ and $s_1(0, 0, t) = 0$ for all t. Let $Y_d(\cdot)$ describe a desired output trajectory; this includes information of the time-derivatives of the desired output, i.e., the desired $\zeta(\cdot)$ represented by $\zeta_d(\cdot)$, and the desired $y^{(r)}(\cdot)$ represented by $y_d^{(r)}(\cdot)$. For this desired output, the input trajectory that maintains exact tracking is given by

$$\begin{aligned} u(t) \triangleq\ & [\beta[\zeta_d(t), \eta(t), t]]^{-1} \\ & \cdot \left[y_d^{(r)}(t) - \alpha[\zeta_d(t), \eta(t), t]\right] \end{aligned} \tag{2}$$

where it is assumed, for invertibility [1], that the absolute value of $\det[\beta]$ is greater than a positive scalar ϵ_β. Then, the system's internal dynamics is given by

$$\begin{aligned} \dot{\eta}(t) =\ & s_1[\zeta_d(t), \eta(t), t] + s_2[\zeta_d(t), \eta(t), t] \\ & \cdot [\beta[\zeta_d(t), \eta(t), t]]^{-1}\left[y_d^{(r)}(t) - \alpha[\zeta_d(t), \eta(t), t]\right] \\ \triangleq\ & s[\eta(t), Y_d(t), t]. \end{aligned} \tag{3}$$

Note that if a bounded solution, $\eta_d(\cdot)$, to the internal dynamics equation (3) is found, then a bounded input trajectory (that maintains exact-tracking) can also be found using (2) as

$$u_d(t) = [\beta[\zeta_d(t), \eta_d(t), t]]^{-1}\left[y_d^{(r)}(t) - \alpha[\zeta_d(t), \eta_d(t), t]\right]. \tag{4}$$

The main difficulty is that the internal dynamics could have an unstable equilibrium at $\eta = 0$. Then, typical solutions to the unstable internal dynamics (3) are unbounded, and consequently the inputs found through (2) are also unbounded. A technique to find bounded solutions to nonlinear unstable internal dynamics has been developed in [1] with extensions made in [4]. Such stable inverse input-state trajectories have been used to develop output-tracking controllers in [3]. In this paper we extend the theory to the case when the internal dynamics is time varying and study computational issues.

II. NOMENCLATURE

If $x(\cdot)$ (also denoted by x) is a vector-valued function with $x(t) = [x_1(t)\ x_2(t)\ x_3(t) \cdots x_n(t)]^T$, i.e., $x(t) \in \Re^n$, then $\|x(t)\|_1 \triangleq \sum_{i=1}^{i=n} |x_i(t)|$ is the standard 1-norm in $\Re^n$, $\|x(t)\|_\infty \triangleq \max_i |x_i(t)|$ is the standard ∞-norm in $\Re^n$ and $\|x(t)\|_{1+\infty} \triangleq \|x(t)\|_1 + \|x(t)\|_\infty$.

If $z(t) \in \Re^{n\times n}$ is a matrix, then $\|z(t)\|_1$ is the induced 1-norm, $\|z(t)\|_1 \triangleq \sup_{y\neq 0,\, y\in\Re^n} \|z(t)y\|_1/\|y\|_1$, $\|z(t)\|_\infty$ is the induced ∞-norm, $\|z(t)\|_\infty \triangleq \sup_{y\neq 0,\, y\in\Re^n} \|z(t)y\|_\infty/\|y\|_\infty$, and $\|z(t)\|_{1+\infty} \triangleq \|z(t)\|_1 + \|z(t)\|_\infty$.

If $x(\cdot)$ (also denoted by x) is a vector-valued measurable function, then $\|x\|_1 \triangleq \|x(\cdot)\|_1 \triangleq \int_{-\infty}^{\infty} \|x(t)\|_1\, dt$, $\|x\|_\infty \triangleq \|x(\cdot)\|_\infty \triangleq \text{ess}\sup_{t\in\Re} \|x(t)\|_\infty$, and $\|x(\cdot)\|_{1+\infty} \triangleq \|x(\cdot)\|_1 + \|x(\cdot)\|_\infty$.

$Y(\cdot) \in \mathcal{B}_r^{n_Y}$ implies $Y(t) \in \Re^{n_Y}$ and $\|Y(\cdot)\|_{1+\infty} < r$.

$\eta(\cdot) \in \mathcal{B}_r^{n_\eta}$ implies $\eta(t) \in \Re^{n_\eta}$ and $\|\eta(\cdot)\|_{1+\infty} < r$.

III. A NONLINEAR INPUT/OUTPUT OPERATOR

In this section, we develop a nonlinear input/output (I/O) operator denoted $\mathcal{N}$, which is central to the inversion of nonlinear nonminimum phase systems. This operator maps bounded inputs into bounded Caratheodory solutions [20] of (3)

$$\dot{\eta}(t) = s(\eta(t), Y_d(t), t), \qquad \eta(\pm\infty) = 0.$$

Note that the input to the operator is $Y_d(\cdot)$, which consists of the desired output trajectory and its time derivatives. The basic idea is to construct a contraction whose fixed point is a solution to (3). The contraction is motivated in the following way. Since it is not known whether $\dot{\eta}(t) = s[\eta(t), Y_d(t), t]$ with $\eta(\pm\infty) = 0$ has a solution, we expand $s(\cdot, \cdot, \cdot)$ into linear and perturbation terms

$$\dot{\eta}(t) = A(t)\eta(t) + [s[\eta(t), Y_d(t), t] - A(t)\eta(t)] \tag{5}$$

where the term in the large square brackets represents the perturbation term. If we know the perturbation term, then we can establish conditions for the existence of a bounded solution to this forced linear system. Our approach is to take a guess at the perturbation term and iterate; we start with

$$\eta_1(\cdot) = 0$$

and at each iteration ($n \geq 1$) solve for a bounded solution to the linear (but potentially unstable) equation

$$\dot{\eta}_{n+1}(t) = A(t)\eta_{n+1}(t) + [s[\eta_n(t), Y_d(t), t] - A(t)\eta_n(t)].$$

We then prove that this iteration converges to $\eta_d(\cdot) \triangleq \mathcal{N}[Y_d(\cdot)]$, a bounded solution of the differential (3). We begin with the linear counterpart of $\mathcal{N}$, denoted $\mathcal{A}$, which finds bounded solutions to the above (unstable) linear equation, i.e.,

$$\eta_{n+1}(t) \triangleq \mathcal{A}[s[\eta_n(\cdot), Y_d(\cdot), \cdot] - A(\cdot)\eta_n(\cdot)](t).$$

A. Linear Operator $\mathcal{A}$

For a system of the form $\dot{\eta}(t) = \hat{A}\eta(t) + \hat{B}u(t)$, with $\hat{A}$ having no $j\omega$-axis eigenvalues, various I/O operators may be defined. The most common operator used in control theory imposes an initial condition of the form $\eta(t_0) = \eta_0$ on the state trajectory. In this subsection, we define an operator $\mathcal{A}$, which imposes an alternative boundary condition, $\eta(\pm\infty) = 0$, on the state trajectory so that the resulting state trajectories are necessarily bounded.

Consider a linear time-varying system of the form

$$\dot{\eta}(t) = A(t)\eta(t) + v(t) \tag{6}$$

where $\eta(t) \in \Re^{n_\eta}$ and $A(t) \in \Re^{n_\eta \times n_\eta}$. The key idea is to make a state transformation splitting (6) into two decoupled subsystems—one of which is exponentially stable and the other is exponentially unstable. By integrating the stable subsystem forward in time and the unstable subsystem backward in time, a bounded solution to the differential equation is obtained (see [3] for a similar approach in the time-invariant case). Although the decoupling of time-invariant linear systems is easily done by using a state-transformation constructed with the eigenvectors of the A matrix, this approach does not lead to the necessary decoupling in the time-varying case [21].

In the following, we use results by Coppel [21] to establish the dichotomic split that enables the extension of the stable-inversion theory to the time-varying case.

Definition 1—Kinematic Similarity [21]: The homogeneous equation

$$\dot{\eta}(t) = A(t)\eta(t) \tag{7}$$

with $A(\cdot)$ continuous for all $t \in \Re^1$ is defined to be *kinematically similar* to

$$\dot{w}(t) = B(t)w(t) \tag{8}$$

provided there exists a transformation $S(\cdot)$ such that for any given solution $w(\cdot)$ to (8)

$$\eta(t) = S(t)w(t) \tag{9}$$

is a solution to (7), $S(t)$ is a continuously-differentiable invertible matrix, and both $S(t)$ and $S^{-1}(t)$ are uniformly bounded for $t \in \Re^1$. □

By substituting (9) into (7), we see that $S(t)$ necessarily satisfies

$$\dot{S}(t) = A(t)S(t) - S(t)B(t).$$

The key is to find a kinematic similarity that achieves the dichotomic split of (6) into stable and unstable subsystems. This dichotomic split is possible provided $A(t)$: 1) is slowly varying in t; 2) is uniformly bounded in t; and 3) is hyperbolic. These conditions are formalized next, and the dichotomic split is established in the following theorem.

Condition I: $A(t) \in \Re^{n_\eta \times n_\eta}$ satisfies Condition I if there exists positive M, α, β such that for every $t \in \Re^1$

1) $\|A(t)\|_{1+\infty} \le M$. See Section II for the definition of $\|\cdot\|_{1+\infty}$;
2) $A(t)$ has k eigenvalues with real part less than or equal to $-\alpha < 0$ and $n_\eta - k$ eigenvalues with real part greater than or equal to $\beta > 0$, where $0 < k < n_\eta$. □

Theorem 1: Let $A(t)$ satisfy Condition I. Then for any positive scalar $\epsilon < \min(\alpha, \beta)$ there exists a positive scalar $\delta = \delta(M, \alpha + \beta, \epsilon)$ such that if $\|(d/dt)A(t)\|_{1+\infty} \le \delta$ for every $t \in \Re^1$, then

1) (7) is *kinematically similar* to

$$\frac{d}{dt}\begin{bmatrix} w_s(t) \\ w_u(t) \end{bmatrix} = \begin{bmatrix} B_s(t) & 0 \\ 0 & B_u(t) \end{bmatrix} \begin{bmatrix} w_s(t) \\ w_u(t) \end{bmatrix} \tag{10}$$

   where $w_s(t) \in \Re^k,\ w_u(t) \in \Re^{(n_\eta - k)}$;

2) $\dot{w}_s(t) = B_s(t)w_s(t)$ is exponentially stable and $\dot{w}_u(t) = B_u(t)w_u(t)$ is exponentially unstable. That is, the respective fundamental matrix solutions (see [20, p. 80]) W_s and W_u satisfy

$$\begin{aligned} \|W_s(t)W_s^{-1}(s)\|_{1+\infty} &\le Ke^{-(\alpha-\epsilon)(t-s)}, && \text{for } t \ge s \\ \|W_u(t)W_u^{-1}(s)\|_{1+\infty} &\le Ke^{-(\beta-\epsilon)(s-t)}, && \text{for } s \ge t; \end{aligned} \tag{11}$$

3) The associated transformation S satisfies

$$\|S(t)\|_{1+\infty} \le N_1(M, \alpha + \beta, \epsilon) \tag{12}$$

$$\|S^{-1}(t)\|_{1+\infty} \le N_2(M, \alpha + \beta, \epsilon). \tag{13}$$

Proof: This follows from Coppel's work [21] (in particular, see Lemma 2 and Theorem 3). □

Remark 1: Note that an important condition in the above theorem is that $A(\cdot)$ is slowly varying in time. □

The above dichotomic split of the system into stable (w_s) and unstable (w_u) subsystems leads to the following bounded solution to (6). Given $A(\cdot)$ satisfying the conditions of Theorem 1, a linear operator $\mathcal{A}$ that finds bounded solutions to (6) is given by the following.

Definition 2: For $v(\cdot) \in L_1 \cap L_\infty$

$$\mathcal{A}(v)(t) \triangleq S(t)\begin{bmatrix} \displaystyle\int_{-\infty}^{t} W_s(t)W_s^{-1}(\tau)S_s(\tau)v(\tau)d\tau \\ \displaystyle -\int_{t}^{\infty} W_u(t)W_u^{-1}(\tau)S_u(\tau)v(\tau)d\tau \end{bmatrix}$$

where

$$\begin{aligned} S_s(t) &\triangleq \begin{bmatrix} I^{k\times k} & 0^{k\times(n_\eta - k)} \end{bmatrix} S^{-1}(t) \\ S_u(t) &\triangleq \begin{bmatrix} 0^{(n_\eta - k)\times k} & I^{(n_\eta - k)\times(n_\eta - k)} \end{bmatrix} S^{-1}(t). \end{aligned}$$

W_s, W_u, S are as in Theorem 1, and without loss of generality $W_s(0) = I^{k\times k}$, $W_u(0) = I^{(n_\eta - k)\times(n_\eta - k)}$. □

Corollary 1.1: Given an operator $\mathcal{A}$ as in Definition 2

1) $\|\mathcal{A}[v(\cdot)](\cdot)\|_{1+\infty} \le \mathbf{G}_{\mathcal{A}}\|v(\cdot)\|_{1+\infty}$ for some $\mathbf{G}_{\mathcal{A}}$ (finite gain property);
2) $\mathcal{A}$: $L_1 \cap L_\infty \to L_1 \cap L_\infty \cap C^0$;
3) $\lim_{t\to-\infty} \mathcal{A}(v)(t) = \lim_{t\to+\infty} \mathcal{A}(v)(t) = 0$.

Proof: See [1]. □

This linear operator, $\mathcal{A}$, finds a bounded solution to an unstable linear system. This operator is extended to a nonlinear operator (that finds bounded solutions to the nonlinear internal dynamics) in the next subsection.

B. Generalizing $\mathcal{A}$ to Nonlinear Case—The Operator $\mathcal{N}$

The following condition requires that the perturbation term in (5) satisfies a locally Lipschitz-like condition in both η and Y.

Condition II: The pair $[s(\cdot, \cdot, \cdot), A(\cdot)]$ satisfies Condition II if for any $Y_1(\cdot), Y_2(\cdot) \in \mathcal{B}_r^{n_Y}$ and $\eta_1(\cdot), \eta_2(\cdot) \in \mathcal{B}_r^{n_\eta}$, the perturbation term $s[\eta(\cdot), Y(\cdot), \cdot] - A(\cdot)\eta(\cdot)$ satisfies the following Lipschitz-like condition (uniformly in t):

$$\begin{aligned} \|\{s[\eta_1(t), Y_1(t), t] - A(t)\eta_1(t)\} \\ - \{s[\eta_2(t), Y_2(t), t] - A(t)\eta_2(t)\}\|_{1+\infty} \\ \le K_1\|\eta_1(t) - \eta_2(t)\|_{1+\infty} + K_2\|Y_1(t) - Y_2(t)\|_{1+\infty} \end{aligned} \tag{14}$$

where s: $\Re^{n_\eta} \times \Re^{n_Y} \times \Re \to \Re^{n_\eta}$, $A \in \Re^{(n_\eta \times n_\eta)}$, and $\mathcal{B}_r$ denotes a ball of radius r in the appropriate space (see Section II). □

Remark 2: Condition II is applicable even when $s(\cdot, \cdot, \cdot)$ is not differentiable. For example, $s[\eta(t), Y(t), t] = \eta(t) + 0.1|\eta(t)| + Y(t)$, which is not differentiable at $\eta = 0$, satisfies the above condition with $A(t) = 1$. However, a $s[\eta(\cdot), Y(\cdot), \cdot]$ with a step discontinuity in the first variable at $\eta = 0$ does not satisfy this Lipschitz-like Condition II for any $A(\cdot)$. □

Next, the linear operator $\mathcal{A}$ is used to define a contraction, $\mathcal{P}_Y(\cdot)$. In particular, Theorem 2 will show that $\mathcal{P}_Y(\cdot)$ is a contraction and Theorem 3 will show that the fixed point of $\mathcal{P}_{Y_d}(\cdot)$ is a bounded solution to the nonlinear internal dynamics (3). Note that for ease of notation, $Y(\cdot)$ and $Y_d(\cdot)$ are represented as Y and Y_d, respectively.

Definition 3:

$$\mathcal{P}_Y(\eta)(t) \triangleq \mathcal{A}[s[\eta(\cdot), Y(\cdot), \cdot] - A(\cdot)\eta(\cdot)](t) \tag{15}$$

where $Y(\cdot) \in \mathcal{B}_r^{n_Y}$, $\eta(\cdot) \in \mathcal{B}_r^{n_\eta}$, $A(\cdot)$ satisfies Condition I and the conditions of Theorem 1, and $[s(\cdot, \cdot, \cdot), A(\cdot)]$ satisfies Condition II. Note that for ease of notation, $\eta(\cdot)$ is represented by η whenever the meaning is clear. $\square$

Theorem 2: Let the conditions in Definition 3 be satisfied and the Lipschitz constants (K_1, K_2) in Condition II satisfy $K_1\mathbf{G}_\mathcal{A} < 1$, and $K_2\mathbf{G}_\mathcal{A} < 1 - K_1\mathbf{G}_\mathcal{A}$, where $\mathbf{G}_\mathcal{A}$ is the bound on the gain of the linear operator $\mathcal{A}$ (see Corollary 1.1). Then, there exists a unique $\eta_Y^*(\cdot) \in \mathcal{B}_r^{n_Y}$, such that $\eta_Y^*(\cdot) = \mathcal{P}_Y[\eta_Y^*(\cdot)](\cdot)$.

Proof: From Corollary 1.1, for any $Y(\cdot) \in \mathcal{B}_r^{n_Y}$ we have

$$\begin{aligned}\|\mathcal{P}_Y(\eta)(\cdot)\|_{1+\infty} &\le \mathbf{G}_\mathcal{A}\, \|s[\eta(\cdot), Y(\cdot), \cdot] - A(\cdot)\eta(\cdot)\|_{1+\infty} \\ &\le \mathbf{G}_\mathcal{A}\, (K_1\|\eta(\cdot)\|_{1+\infty} + K_2\|Y(\cdot)\|_{1+\infty}).\end{aligned} \tag{16}$$

Since $Y(\cdot) \in \mathcal{B}_r^{n_Y}$ implies $\|Y(\cdot)\|_{1+\infty} < r$, we have from $K_2\mathbf{G}_\mathcal{A} < 1 - K_1\mathbf{G}_\mathcal{A}$ that $\mathcal{P}_Y(\cdot)$: $\mathcal{B}_r^{n_\eta} \to \mathcal{B}_r^{n_\eta}$. Next, from the definition of $\mathcal{P}_Y$, linearity of $\mathcal{A}$, Corollary 1.1, and Condition II, we obtain

$$\|\mathcal{P}_Y(\eta_1)(\cdot) - \mathcal{P}_Y(\eta_2)(\cdot)\|_{1+\infty} \le \mathbf{G}_\mathcal{A}\, K_1\, \|\eta_1(\cdot) - \eta_2(\cdot)\|_{1+\infty}. \tag{17}$$

$\mathbf{G}_\mathcal{A}K_1 < 1$ implies that $\mathcal{P}_Y(\cdot)$ is a contraction, and the theorem follows from the Contraction Mapping theorem. $\square$

Remark 3: Equation (17) implies that $\mathcal{P}_Y[\eta(\cdot)](\cdot)$ is Lipschitz in $\eta(\cdot)$. $\square$

Theorem 3: Let the conditions of Theorem 2 be satisfied, $Y(\cdot) \in \mathcal{B}_r^{n_Y}$, and $\mathcal{N}(Y) \triangleq \eta_Y^*(\cdot)$. Then (see [3] for an analogous result)

1) $\mathcal{N}(\cdot)$: $\mathcal{B}_r^{n_Y} \to \mathcal{B}_r^{n_\eta} \cap C_0$ and $(d/dt)[\mathcal{N}(Y)](t) = s[[\mathcal{N}(Y)](t), Y(t), t]$ a.e. in $t \in \Re$;

2) $\lim_{t\to-\infty} [\mathcal{N}(Y)](t) = \lim_{t\to+\infty} [\mathcal{N}(Y)](t) = 0$.

Proof: Theorem 2 and the Contraction Mapping theorem imply the existence of a unique fixed point $\eta_Y^*(\cdot) \in \mathcal{B}_r^{n_\eta}$ such that

$$\begin{aligned}\eta_Y^*(t) &= \mathcal{P}_Y(\eta_Y^*)(t) \\ &= \mathcal{A}\,[s[\eta_Y^*(\cdot), Y(\cdot), \cdot] - A(\cdot)\eta_Y^*(\cdot)](t)\end{aligned}$$

$$\|\eta_Y^*(\cdot)\|_{1+\infty} \le \frac{K_2\mathbf{G}_\mathcal{A}}{1 - K_1\mathbf{G}_\mathcal{A}}\|Y(\cdot)\|_{1+\infty}. \tag{18}$$

Hence, $\eta_Y^*(\cdot) \in L_1 \cap L_\infty \cap C^0$ from which the first assertion of the theorem follows. Next, consider the sequence

$$\begin{aligned}\eta_Y^0(\cdot) &= 0 \\ \eta_Y^{n+1}(t) &= \mathcal{P}_Y(\eta_Y^n)(t)\end{aligned} \tag{19}$$

which converges to $\eta_Y^*(\cdot)$ uniformly in t. The uniform convergence of this sequence, and the fact that

$$\lim_{t\to-\infty} \eta_Y^n(t) = \lim_{t\to+\infty} \eta_Y^n(t) = 0$$

for all $n > 0$ (by Corollary 1.1), implies the second assertion of the theorem. $\square$

In particular, the last theorem shows that $\eta_d(\cdot) \triangleq \mathcal{N}(Y_d)](\cdot) = \eta_{Y_d}^*(\cdot)$ is a bounded solution to the internal dynamics (3), i.e., $\dot\eta_d(t) = s[\eta_d(t), Y_d(t), t]$.

IV. COMPUTATIONAL ISSUES

Computation of the inverse input trajectory, $u_d(\cdot)$ requires: 1) iterative integrations over an infinite time window to find $[\mathcal{N}(Y_d)](\cdot)$ (Section III) and 2) computation of a transformation S that achieves the dichotomic split (Definition 1), which is also an iterative process (see [21, Th. 2 and 3]). In this section, we show that it is possible to compute with a single iterative process, an *approximate* inverse input-trajectory $\hat u_d(\cdot)$ and establish bounds on the error $\|\hat u_d(\cdot) - u_d(\cdot)\|_{1+\infty}$.

A. Errors Due to Truncations and Finite Iterations

In the last section, we found a bounded solution to the internal dynamics (3) by iteratively finding the fixed point of the contraction $\mathcal{P}_{Y_d}(\cdot)$. Each step in the iterative procedure required computations over the whole real line. Here, we truncate $\mathcal{P}_{Y_d}(\cdot)$ to the compact interval $[-T, T]$ and, thus, define a new operator $\mathcal{P}_{Y_d, T}(\cdot)$. In the following, we begin with a general output trajectory $Y(\cdot)$.

Definition 4: Let the conditions in Definition 3 be satisfied, $v(\cdot) \in L_1 \cap L_\infty$, $Y(\cdot) \in \mathcal{B}_r^{n_Y}$, and $T \in \Re$. Then

$$\begin{aligned}\mathcal{A}_T(v)(t) &\triangleq S(t)\begin{bmatrix}\displaystyle\int_{-T}^{t} W_s(t)W_s^{-1}(\tau)\, S_s(\tau)\, v(\tau)\, d\tau \\ \displaystyle -\int_{t}^{T} W_u(t)W_u^{-1}(\tau)\, S_u(\tau)\, v(\tau)\, d\tau\end{bmatrix}, \\ &\qquad \text{for all } \ t \in [-T, T] \\ &\triangleq 0 \qquad \text{otherwise,} \\ \mathcal{P}_{Y, T}(\eta)(t) &\triangleq \mathcal{A}_T[s[\eta(\cdot), Y(\cdot), \cdot] - A(\cdot)\eta(\cdot)](t)\end{aligned}$$

with S, S_s, S_u, W_s, W_u as in Definition 2. $\square$

The following theorem will establish that the above truncated map, $\mathcal{P}_{Y, T}(\cdot)$, is also a contraction. The goal is to show that for a given $Y(\cdot)$, the fixed point of the truncated map is *close* to the fixed point of the original map $\mathcal{P}_Y(\cdot)$.

Theorem 4: If $\mathbf{G}_\mathcal{A}$ and the bound on the gain of the linear operator $\mathcal{A}$ K_1 and K_2 are sufficiently small, then there exists $\eta_{Y,,T}^*(\cdot)$, a unique fixed point of $\mathcal{P}_{Y, T}(\cdot)$. Further, the sequence $\{\eta_{Y, T, m}(\cdot)\}_{m=0}^\infty$, defined by

$$\begin{aligned}\eta_{Y, T, 0}(\cdot) &\triangleq 0 \\ \eta_{Y, T, m+1}(t) &\triangleq \mathcal{P}_{Y,T}(\eta_{Y, T, m})(t)\end{aligned} \tag{20}$$

converges to $\eta_{Y,T}^*(\cdot)$ in the $\|\cdot\|_{1+\infty}$ sense (as m increases).

Proof: This follows from arguments similar to those in the proof of Theorem 3. $\square$

The following lemma establishes that $\mathcal{P}_Y(\eta)(\cdot) = \lim_{T\to\infty} \mathcal{P}_{Y, T}(\eta)(\cdot)$, in the $\|\cdot\|_{1+\infty}$ sense.

Lemma 1: Let $\eta(\cdot) \in \mathcal{B}_r^{n_\eta}$. Then, $\lim_{T\to\infty} \|\mathcal{P}_Y(\eta)(\cdot) - \mathcal{P}_{Y, T}(\eta)(\cdot)\|_{1+\infty} = 0$.

Proof: We first show that

$$\lim_{T\to\infty} \|\mathcal{P}_Y(\eta)(\cdot) - \mathcal{P}_{Y, T}(\eta)(\cdot)\|_1 = 0.$$

For ease in terminology we will use the following notations:

$$\begin{aligned}\gamma(t) &\triangleq s[\eta(t), Y(t), t] - A(t)\eta(t) \\ \gamma_s(t, \tau) &\triangleq W_s(t)W_s^{-1}(\tau)\, S_s(\tau)\, \gamma(\tau) \\ \gamma_u(t, \tau) &\triangleq W_u(t)W_u^{-1}(\tau)\, S_u(\tau)\, \gamma(\tau).\end{aligned}$$

Then, from Definitions 2–4, we obtain

$$\begin{aligned}&\|\mathcal{P}_Y(\eta)(\cdot) - \mathcal{P}_{Y,\,T}(\eta)(\cdot)\|_1 \\ &\quad= \int_{-\infty}^{-T} \|\mathcal{A}(\gamma)(t) - \mathcal{A}_T(\gamma)(t)\|_1\,dt \\ &\qquad+ \int_{-T}^{T} \|\mathcal{A}(\gamma)(t) - \mathcal{A}_T(\gamma)(t)\|_1\,dt \\ &\qquad+ \int_{T}^{\infty} \|\mathcal{A}(\gamma)(t) - \mathcal{A}_T(\gamma)(t)\|_1\,dt.\end{aligned}$$

We illustrate the proof technique for only one of the terms. The rest of the proof follows from similar algebraic manipulations. From Definitions 2 and 4

$$\begin{aligned}&\int_{-\infty}^{-T} \|\mathcal{A}(\gamma)(t) - \mathcal{A}_T(\gamma)(t)\|_1\,(dt) \\ &\quad\le \int_{-\infty}^{-T} \|S(t)\|_1 \int_{-\infty}^{t} \|\gamma_s(t,\,\tau)\|_1\,d\tau\,dt \\ &\qquad+ \int_{-\infty}^{-T} \|S(t)\|_1 \int_{T}^{+\infty} \|\gamma_u(t,\,\tau)\|_1\,d\tau\,dt.\end{aligned}$$

For the first term on the right-hand side (r.h.s.) apply (11)–(13) to obtain

$$\begin{aligned}&\int_{-\infty}^{-T} \|S(t)\|_1 \int_{-\infty}^{t} \|\gamma_s(t,\,\tau)\|_1\,d\tau\,dt \\ &\quad\le N_1N_2K \int_{-\infty}^{-T}\int_{-\infty}^{t} e^{-(\alpha-\epsilon)(t-\tau)}\|\gamma(\tau)\|_1\,d\tau\,dt \\ &\quad\le N_1N_2K \int_{-\infty}^{-T}\int_{\tau}^{-T} e^{-(\alpha-\epsilon)(t-\tau)}\|\gamma(\tau)\|_1\,dt\,d\tau \\ &\quad= \frac{N_1N_2K}{\alpha-\epsilon}\int_{-\infty}^{-T}\left[1 - e^{(\alpha-\epsilon)(T+\tau)}\right]\|\gamma(\tau)\|_1\,d\tau.\end{aligned}$$

Since $\tau \le -T$ and $\alpha - \epsilon > 0$, we have $0 < e^{(\alpha-\epsilon)(T+\tau)} \le 1$ and hence

$$\int_{-\infty}^{-T} \|S(t)\|_1 \int_{-\infty}^{t} \|\gamma_s(t,\,\tau)\|_1 d\tau\,dt \le \frac{N_1N_2K}{\alpha-\epsilon}\int_{-\infty}^{-T}\|\gamma(\tau)\|_1 d\tau.$$

This tends to zero as $T \to \infty$ if $\gamma(\cdot) \in L_1$, which follows from the definition of $\gamma(\cdot)$, Condition II, $\eta(\cdot) \in L_1 \cap L_\infty$, and $Y(\cdot) \in L_1 \cap L_\infty$. Similarly, the other terms also tend to zero as $T \to \infty$. The key is to rewrite them, either as integrals from $-\infty$ to $-T$ or as integrals from T to ∞ and then show that the integrals go to zero as T tends to infinity.

Next, we show that $\lim_{T\to\infty} \|\mathcal{P}_Y(\eta)(t) - \mathcal{P}_{Y,\,T}(\eta)(t)\|_\infty = 0$, uniformly in t. We split the proof into three parts: 1) $t \le -T$; 2) $-T \le t \le T$; and 3) $T \le t$. We illustrate the proof technique for case $t \le -T$ only. For $t \le -T$

$$\begin{aligned}&\|\mathcal{P}_Y(\eta)(t) - \mathcal{P}_{Y,\,T}(\eta)(t)\|_\infty \\ &\quad\le \|\mathcal{P}_Y(\eta)(t) - \mathcal{P}_{Y,\,T}(\eta)(t)\|_1 \\ &\quad= \|\mathcal{A}(\gamma)(t) - \mathcal{A}_T(\gamma)(t)\|_1 \\ &\quad\le \|S(t)\|_{1+\infty}\Bigg(\int_{-\infty}^{t} \|\gamma_s(t,\,\tau)\,d\tau\|_1 \\ &\qquad\qquad+ \int_{T}^{\infty} \|\gamma_u(t,\,\tau)\,d\tau\|_1\Bigg) \\ &\quad\le N_1N_2K\Bigg(\int_{-\infty}^{t} e^{-(\alpha-\epsilon)(t-\tau)}\|\gamma(\tau)\,d\tau\|_1 \\ &\qquad\qquad+ \int_{T}^{\infty} e^{-(\beta-\epsilon)(\tau-t)}\|\gamma(\tau)\,d\tau\|_1\Bigg) \\ &\quad\le N_1N_2K\Bigg(\int_{-\infty}^{-T} \|\gamma(\tau)\,d\tau\|_1 \\ &\qquad\qquad+ \int_{T}^{\infty} \|\gamma(\tau)\,d\tau\|_1\Bigg).\end{aligned}$$

$\gamma \in L_1$ implies that the r.h.s. tends to zero as $T \to \infty$ independent of t, and hence the left-hand side (l.h.s.) tends to zero uniformly in t. The other two cases, when $t \ge -T$, can be proved by similar arguments.

Thus, the limit is established in the $\|\cdot\|_1$ and in the $\|\cdot\|_\infty$ norms, which completes the proof. □

The next lemma states that $\eta^*_{Y,\,T}(\cdot)$, the fixed point of the truncated operator $\mathcal{P}_{Y,\,T}(\cdot)$, tends to $\eta^*_Y(\cdot)$, the fixed point of the operator $\mathcal{P}_Y(\cdot)$, as $T \to \infty$ (see [20, p. 7] for a related theorem).

Lemma 2: For all $\epsilon_1 > 0$, there exists $T_1(\epsilon_1)$ such that $T > T_1(\epsilon_1)$ implies that $\|\eta^*_{Y,\,T}(\cdot) - \eta^*_Y(\cdot)\|_{1+\infty} \le \epsilon_1$.

Proof:

$$\begin{aligned}&\|\eta^*_{Y,\,T}(\cdot) - \eta^*_Y(\cdot)\|_{1+\infty} \\ &\quad= \|\mathcal{P}_{Y,\,T}\big(\eta^*_{Y,\,T}\big)(\cdot) - \mathcal{P}_Y(\eta^*_Y)(\cdot)\|_{1+\infty} \\ &\quad\le \|\mathcal{P}_Y\big(\eta^*_{Y,\,T}\big)(\cdot) - \mathcal{P}_Y(\eta^*_Y)(\cdot)\|_{1+\infty} \\ &\qquad+ \|\mathcal{P}_Y\big(\eta^*_{Y,\,T}\big)(\cdot) - \mathcal{P}_{Y,\,T}\big(\eta^*_{Y,\,T}\big)(\cdot)\|_{1+\infty}\end{aligned}$$

using the triangle inequality. Next, using the Lipschitz property of $\mathcal{P}_Y(\cdot)$ we obtain (see Remark 3)

$$\begin{aligned}&\|\eta^*_{Y,\,T}(\cdot) - \eta^*_Y(\cdot)\|_{1+\infty} \\ &\quad\le K_1\mathbf{G}_\mathcal{A}\|\eta^*_{Y,\,T}(\cdot) - \eta^*_Y(\cdot)\|_{1+\infty} \\ &\qquad+ \|\mathcal{P}_Y\big(\eta^*_{Y,\,T}\big)(\cdot) - \mathcal{P}_{Y,\,T}\big(\eta^*_{Y,\,T}\big)(\cdot)\|_{1+\infty} \\ &\Longrightarrow\ \|\eta^*_{Y,\,T}(\cdot) - \eta^*_Y(\cdot)\|_{1+\infty} \\ &\quad\le \frac{1}{1 - K_1\mathbf{G}_\mathcal{A}}\|\mathcal{P}_Y\big(\eta^*_{Y,\,T}\big)(\cdot) - \mathcal{P}_{Y,\,T}\big(\eta^*_{Y,\,T}\big)(\cdot)\|_{1+\infty}.\end{aligned}$$

Note, from Lemma 1, we have $\mathcal{P}_Y(\eta^*_{Y,\,T}) = \lim_{T\to\infty}\mathcal{P}_{Y,\,T}(\eta^*_{Y,\,T})$ and, hence, the r.h.s. can be made arbitrarily small by choosing T large enough. □

The next theorem gives the main result that the inverse trajectory can be approximated (arbitrarily closely) by choosing a large enough time window for computations in each iteration and by using a sufficiently large number of iterations.

Theorem 5: Given $\epsilon > 0$, there exists M, T^* such that $m > M, T > T^*$ implies that $\|\hat{u}_d(\cdot) - u_d(\cdot)\|_{1+\infty} \le \epsilon$, where u_d is defined by (4) and the approximate inverse-input $\hat{u}_d(\cdot)$ is defined as

$$\begin{aligned}\hat{u}_d(t) \triangleq\ &[\beta[\zeta_d(t),\,\eta_{Y_d,\,T,\,m}(t),\,t]]^{-1} \\ &\cdot\Big[y_d^{(r)}(t) - \alpha[Y_d(t),\,\eta_{Y_d,\,T,\,m}(t),\,t]\Big].\end{aligned}\tag{21}$$

Proof: Lemma 2 and the convergence of sequence $\eta_{Y_d,\,T,\,m}(\cdot)$ [see (20)] imply that $\|\eta_{Y_d,\,T,\,m}(\cdot) - \eta^*_{Y_d}(\cdot)\|_{1+\infty}$ can be made arbitrarily small by choosing T and m large enough. The theorem follows from the continuity of $\hat{u}(t)$ in $\eta(t)$. □

Summarizing, given an $\epsilon > 0$, $\hat{u}_d(\cdot)$ can be computed through a finite number (m) of iterative integrations performed over a closed time interval $[-T^*,\,T^*]$, such that $\|\hat{u}_d(\cdot) - u_d(\cdot)\|_{1+\infty} < \epsilon$. This bound ϵ can be made arbitrarily small by increasing T^* and m.

B. Computation of $S(\cdot)$

Given a nonlinear time-varying internal dynamics, $\dot{\eta}(t) = s[\eta(t),\,Y(t),\,t]$, the key is to find a pair $(A(\cdot),\,S(\cdot))$ such that: 1) $s[\eta(t),\,Y(t),\,t] - A\eta(t)$ satisfies the Lipschitz-like Condition II and 2) the change of variables $\eta(t) = S(t)w(t)$ achieves the dichotomic split of the linear equation $\dot{\eta}(t) = A(t)\eta(t)$. The existence of the block-diagonalizing transformation $S(t)$ has been studied in [21]; however, the computation of the transformation is iterative. Below, we present a modified algorithm that circumvents the iteration.

Algorithm for $S(\cdot)$ (Coppel):

1) Choose $\hat{A}(\cdot)$ (not necessarily equal to $(d/d\eta)s$) such that: a) $\|\hat{A}(t)\|_{1+\infty}$ is bounded and slowly varying and b) $s[\eta(t), Y(t), t] - \hat{A}\eta(t)$ satisfies the Lipschitz-like Condition II.

2) Compute the projection operator (onto the stable subspace of $\hat{A}(t)$)

$$P(t) \triangleq \frac{1}{2\pi i} \int_{\gamma} [\lambda I - \hat{A}(t)]^{-1} \, d\lambda$$

where γ is the simple closed curve in the left half-plane formed by the imaginary axis and part of the circle $|\lambda| > \|\hat{A}\|$ (see [21, p. 511]).

3) Solve the first-order linear ordinary differential equation ([21, p. 513])

$$\dot{U}(t) = [\dot{P}(t)P(t) - P(t)\dot{P}(t)]U(t), \qquad U(0) = I.$$

4) Compute $S(t) \triangleq U(t)R^{-1}(t)$ with

$$R(t) \triangleq [P(0)U^{*}(t)U(t)P(0) + \{I - P(0)\}U^{*}(t)U(t)\{I - P(0)\}]^{1/2}.$$

This $S(t)$ cannot be used to decouple $\hat{A}(t)$, although such a decoupling transformation can be found using the iterative algorithm in [21, Th. 2]. Instead, we choose an alternate $A(\cdot)$ matrix.

5) Choose $A(t) \triangleq \hat{A}(t) + \dot{S}(t)S^{-1}(t)$. □

It follows from Coppel [21, Th. 3] that: 1) $S(\cdot)$ decomposes $A(\cdot)$ as in Theorem 1 and 2) $\|\dot{S}(t)S^{-1}(t)\|_{1+\infty}$ is proportional to the $\|\hat{A}(\cdot)\|_{1+\infty}$. This proportional dependence implies that $A(\cdot)$ also satisfies the Lipschitz-like Condition II [if it is satisfied by a sufficiently slowly varying $\hat{A(\cdot)}$]. This concludes the algorithm to compute the dichotomic split of the linearized internal dynamics.

V. CONCLUSIONS

In this paper, we have defined a new method for inverting nonlinear nonminimum-phase time-varying systems and have presented a constructive algorithm for computing inverse trajectories. The inverse trajectories form the basis of a new exact output tracking controller. Since the noncausal inverses decay to zero exponentially in negative time, truncation is attractive and was analyzed; all the desirable continuity properties of the truncation were shown to hold.

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