stochastic stability framework, stability of all operation modes is not even required.

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Nonlinear Repetitive Control

Jayati Ghosh and Brad Paden

***Abstract*—Repetitive controllers are generally applied to reject periodic disturbances and to track periodic reference signals with a known period. Their design is based on *The Internal Model Principle*, proposed by Francis and Wonham. This paper describes a new finite-dimensional SISO repetitive controller for two different classes of nonlinear plants. Simulation results show asymptotic tracking of the periodic reference signal by the proposed repetitive controller in closed loop up to the Nth harmonic frequency. A proof of robustness of the repetitive control system to small nonlinearities, like actuator nonlinearities, is provided.**

***Index Terms*—Internal model principle, nonlinear, tracking.**

I. INTRODUCTION

Periodic signals commonly occur in robotics, servo mechanisms, and other similar tracking scenarios, either in form of reference inputs or disturbances. In linear time-invariant plants, repetitive control builds on the well-known internal model principle [1]–[3] to provide exact asymptotic output tracking of periodic inputs. The internal model principle states that the output of a plant can be made to asymptotically track a class of reference commands without a steady-state error if the generator for the reference signal is included in the stable closed-loop system. If a periodic input signal has a finite Fourier series, then a finite number of internal models (one for each harmonic) can be used to produce asymptotic tracking. Likewise, if the periodic input has an infinite Fourier series, an infinite number of controller models (i.e., $j\omega$-axis poles) are required for exact tracking. Fortunately, a simple delay line can be used to produce an infinite number of poles, but the system dynamics are, nonetheless, infinite dimensional. Such periodic tracking problems are surprisingly common. For example, every computer disk drive uses some type of linear repetitive control to compensate for repeatable runout in the disk bearings. Other applications of significant economic value include eccentricity compensation in rolling mills, noncircular machining of pistons and camshafts, AFM control, and optical turning.

The innovation of repetitive control was motivated by a power supply regulation problem and is due to Inoue *et al.* [4]. Early progress was made in papers by Nakano, Iwai, Omata, and Hara [5], [6], culminating in a seminal paper on the stability of linear infinite-dimensional repetitive controllers [7]. Repetitive control theory became more accessible with the appealing discrete-time formulation of Tomizuka *et al.* [8]; the discrete-time formulation was developed further to cover robustness analysis [9]. Disturbance rejection is a particularly important problem in repetitive control, and this has been addressed in the context of the discrete-time formulation for disk-drive shock disturbance rejection [10]. The discrete-time formulation also allows segments of a periodic reference input to be selected for exact tracking, thus saving computer memory [11]. The repetitive control theory community joined forces with the multivariable control community beginning with the work of Ledwich and Bolton [12]. Further research on linear repetitive control design is due to Güvenc [13] where sensitivity minimization at discrete points is used to design a repetitive controller. Roh and Chung

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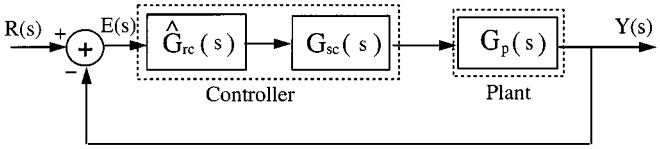

Fig. 1. Repetitive controller: continuous-time model.

have a nice robustness result employing Kharitonov's theorem [14], and Sadegh [15] has devised a method using feedforward.

On the topic of nonlinear repetitive control, there has been very limited research effort. A paper by Omata *et al.* [16] establishes abstract passivity bounds on the nonlinearity such that the system remains L_2 stable and the error signal is L_2. Another paper by Hikita *et al.* [17] uses sliding modes to eliminate nonlinearity, thus reducing the problem to a linear one. An analysis of higher order internal models for nonlinear plants has appeared in Huang *et al.* [26]. Independent results by Khalil [27], similar to our approach [24] have also recently appeared. The concept of kth-order tracking in [26] differs from our definition. Huang and Lin use this term to describe a steady-state error that is infinite-simal of order k with respect to the amplitude of disturbance input.

In this paper, we apply repetitive controllers to a nonlinear plant and analyze the tracking performance. In Section II, a new finite-dimensional, continuous-time repetitive controller is proposed. In Section III, we consider repetitive control of a nonlinear system of the form $\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})u$; $y = h(\boldsymbol{x})$, which has a well-defined relative degree. In Section IV, we present an example of output tracking with the repetitive controller for a nonlinear system, which does not have a well-defined relative degree. The method is applied here in the context of a common laboratory system—the ball and beam system—where the repetitive controller achieves the best periodic tracking results to date for this system [18], [19]. Finally, Section V concludes the paper.

II. Model of a Continuous-Time Repetitive Control System

Before we proceed further, consider the block diagram of a SISO continuous-time repetitive control system shown in Fig. 1. Here, $G_p(s)$ is the transfer function of the linear plant, $\hat{G}_{\rm rc}(s)$ is the proposed repetitive controller containing an internal model of the desired output, and $G_{\rm sc}(s)$ is an observer-based controller, designed to stabilize the closed-loop system. The repetitive controller is ideally given by the following transfer function: $G_{\rm rc}(s) = (1/1 - e^{-sT})$, which has its poles at $\pm jn\omega$, where $n \in \mathbb{Z}$, the set of integers and T is the time period of the periodic reference signal $r(t)$. The above infinite order model is difficult to realize in practice, however, and is not robust. Therefore, we propose approximating it by a finite order model given by

$$\hat{G}_{\rm rc}(s) = \frac{1}{s\prod_{n=1}^{N}(s^2 + n^2\omega^2)} \tag{1}$$

with poles only at locations $jn\omega$, where $n = -N, -N + 1, \cdots, 0, \cdots, N$, where N is a finite integer. We prove in Section IV (Theorem 1 and Proposition 1) that the tracking error $e(t)$ has zero harmonic content at the repetitive frequency and its harmonics up to N. Also, we show (Proposition 2) that the bound on the tracking error decreases with N (where N is the number of oscillators modeled in the controller). This justifies the fact that in the limit $N \to \infty$, the above model of the repetitive controller works as well as the ideal model in achieving perfect tracking for typical reference signals.

III. Repetitive Control of I/O Feedback Linearized Systems

The availability of powerful methods for solving linear control problems has motivated the investigation of various procedures for linearizing nonlinear systems. If the plant in Fig. 1 is nonlinear, but is input–output (I/O) feedback linearizable [20], it will still be possible to achieve asymptotic tracking with the repetitive control system shown in Fig. 1. The well-known work of Isidori [20] and others has established conditions under which a nonlinear state equation with a specified output can be transformed, via state feedback and state coordinates change, into a *normal form* [20], which has the property of being I/O linear. Consider now the commonly studied nonlinear plant

$$\begin{aligned}\dot{\boldsymbol{x}} &= f(\boldsymbol{x}) + g(\boldsymbol{x})u \\ y &= h(\boldsymbol{x})\end{aligned} \tag{2}$$

where the functions f, g, and h are sufficiently smooth.

In this section, the tracking problem for a class of nonlinear systems that are I/O feedback linearizable (i.e., have a well-defined relative degree at $\boldsymbol{x} = \boldsymbol{x}^o$) are studied, and the conditions under which a repetitive controller can be used to exactly reproduce the prescribed reference output of the plant are discussed. Consider a representative *third* order nonlinear system from this class, which is characterized in *normal form* by

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \\ \dot{\eta} &= -\eta^3 + \Psi(\eta, \xi) \\ y &= \xi_1.\end{aligned} \tag{3}$$

The block diagram of the closed-loop system with a repetitive controller of order $N = 7$ is given in Fig. 2. The ξ-subsystem is stabilized by the control law $u = -4\xi_1 - 4\xi_2$. It is to be noted that the prime concern in the design of the control laws is that the variables representing the internal behavior of the system remain *bounded* when a specific control law is imposed. Therefore, instead of relying on the convergence of $\eta(t)$ to zero, we can strengthen the input-to-state stability property of the η-subsystem, $\dot{\eta} = f(\eta) + \Psi(\eta, \xi)$, by requiring that for any bounded input $\xi(t)$, the corresponding solution $\eta(t)$ of the η-subsystem be bounded.

We now present the simulation results for two different interconnection terms $\Psi(\eta, \xi)$. Fig. 3 shows the results obtained with sinusoidal reference input and $\Psi(\eta, \xi) = \xi_2\eta^3$. In this case, $\eta(t)$ decays to zero as $t \to \infty$ and asymptotic tracking is achieved. Fig. 4 shows the result when the interconnection term $\Psi(\eta, \xi)$ is given by $\xi_2\eta$, and with the same reference input. Here, too, asymptotic tracking of the periodic reference signal is achieved, although $\eta(t)$ is now periodic (i.e., bounded, but does not converge to zero).

IV. Repetitive Control of Singular Nonlinear Systems

It is not unlikely for a nonlinear plant given by (2) to have points in its state space where the relative degree cannot be defined. In general, the conditions for feedback linearization of a nonlinear system with specified output are restrictive. Therefore, it is of practical interest to investigate situations in which these conditions fail but do so only *slightly.* Following the approach of Hauser *et al.* [18], we begin with an example drawn from control laboratories—the ball and beam experiment—and show how the exact input–output linearization approach may fail.

The ball and beam system is depicted in Fig. 5. The control objective is to control the torque τ applied at the pivot of the beam, such that the ball can roll on the beam and track a desired trajectory. Let the moment of inertia of the beam be J; the mass and moment of inertia of the ball be M and J_b, respectively; the radius of the ball be R; and

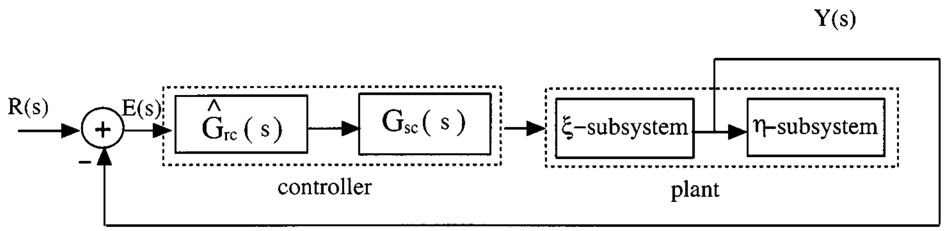

Fig. 2. Output tracking: feedback-linearized model.

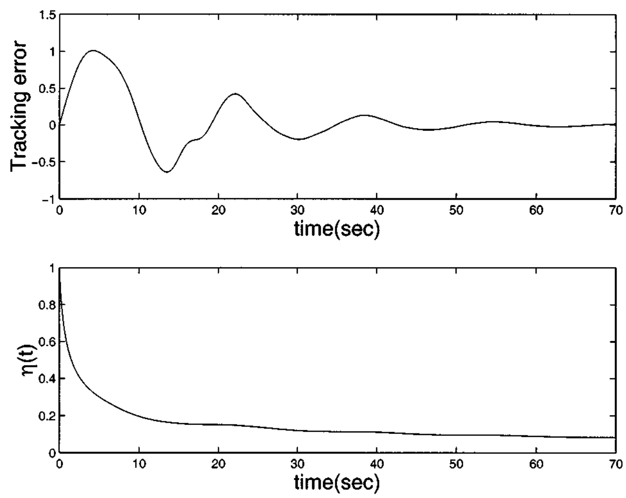

Fig. 3. I/O feedback linearized model.

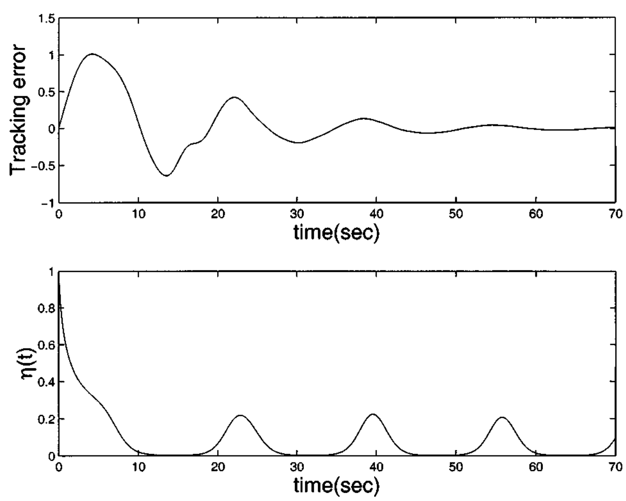

Fig. 4. I/O feedback linearized model; $\eta(t)$ is periodic.

the acceleration of gravity be G. Choosing the beam angle θ and the ball position r as generalized position coordinates for the system, the equations for motion are given by

$$\left(\frac{J_b}{R^2}+M\right)\ddot{r}+MG\sin\theta-Mr\dot{\theta}^2=0$$
$$\left(Mr^2+J+J_b\right)\ddot{\theta}+2Mr\dot{r}x\dot{\theta}+MGr\cos\theta=\tau$$

where τ is the torque applied to the beam. Defining $B := M/(J_b/R^2+M)$ and changing the coordinates in the input space from τ to u using the invertible nonlinear transformation

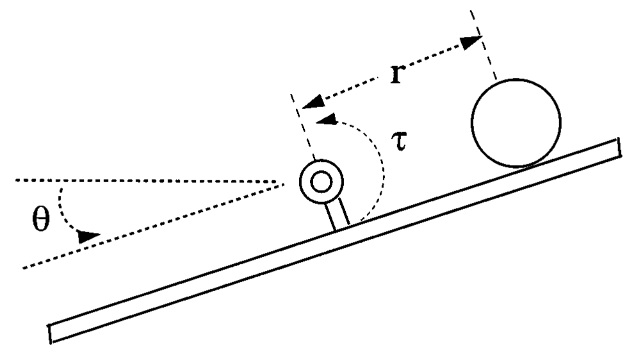

Fig. 5. The ball and beam system.

$\tau = 2Mr\dot{r}\dot{\theta}+MGr\cos\theta+(Mr^2+J+J_b)u$, the system can be written in state-space form as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \underbrace{\begin{bmatrix}x_2\\ B(x_1x_4^2-G\sin x_3)\\ x_4\\ 0\end{bmatrix}}_{f(\boldsymbol{x})} + \underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}_{g(\boldsymbol{x})}u \tag{4}$$

$$y = \underbrace{x_1}_{h(\boldsymbol{x})} \tag{5}$$

where $\boldsymbol{x}=(x_1, x_2, x_3, x_4)^T := (r, \dot{r}, \theta, \dot{\theta})^T$ and $y=h(\boldsymbol{x}) := r$. The output $y(t)$ is to asymptotically track a desired trajectory $y_d(t)$, i.e., $y(t)\to y_d(t)$ as $t\to\infty$. We successively differentiate the output y three times until the input u appears algebraically for the first time in $y^{(3)} = Bx_2x_4^2 - BGx_4\cos x_3 + 2Bx_1x_4u$. Note, however, that the coefficient of u is zero whenever $\dot{\theta}$ or r is zero. It follows, therefore, that the relative degree of this system is not well defined at the point of interest for the given output. Thus, the exact I/O linearization approach is not applicable to this problem.

We now briefly review the derivation of the approximate I/O linearized model [18] for convenience. Since the system fails to have a well-defined relative degree because of the centrifugal acceleration term $Bx_1x_4^2$, we design our *approximate system* by simply neglecting it. A nonlinear change of coordinates $\xi=\Phi(\boldsymbol{x})$ is constructed to transform the system as follows:

$$\begin{aligned}
\xi_1 &= y = \Phi_1(\boldsymbol{x})\\
\dot{\xi}_1 &= \underbrace{x_2}_{\xi_2=\Phi_2(\boldsymbol{x})}\\
\dot{\xi}_2 &= \underbrace{-BG\sin x_3}_{\xi_3=\Phi_3(x)} + \underbrace{Bx_1x_4^2}_{\Psi_2(\boldsymbol{x})(\text{the annoying term!})}\\
\dot{\xi}_3 &= \underbrace{-BGx_4\cos x_3}_{\xi_4=\Phi_4(\boldsymbol{x})}\\
\dot{\xi}_4 &= \underbrace{BGx_4^2\sin x_3}_{b(x)} + \underbrace{(-BG\cos x_3)}_{a(\boldsymbol{x})}u.
\end{aligned} \tag{6}$$

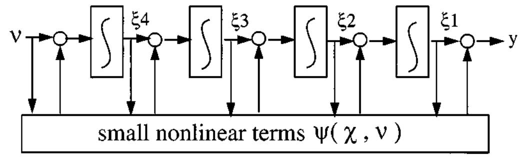

Fig. 6. Approximate I/O linearization: a chain of integrators perturbed by small nonlinear terms.

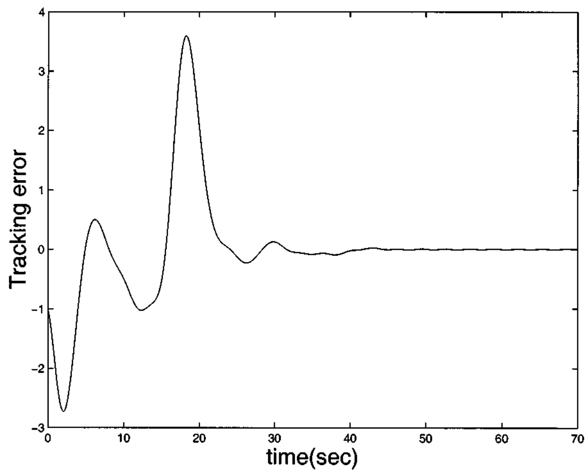

Fig. 7. Tracking error: ball and beam problem.

When a state feedback law $u(\boldsymbol{x}, v) = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})v$ is applied, the system resembles a chain of integrators perturbed by higher order terms $\Psi(\boldsymbol{x}, v)$, as depicted in Fig. 6. The first component of ξ is the system output y. Each successive component of the coordinate change is constructed by differentiating the previous component along the system trajectories and discarding the undesirable higher order terms.

As the objective is to make the output track the prescribed periodic reference trajectory such as $y_d = \sin \omega t$, we design a repetitive control for the ball and beam system given by (6). Fig. 7 shows a surprising result; the tracking error achieved by the repetitive control system when a sinusoid is applied as a reference signal is essentially zero. The observer-based stabilizing controller is constructed based on the linear component of the model (i.e., neglecting the nonlinear terms of the approximate I/O linearized system). Noting that the nonlinear term $Bx_1x_4^2$ is generally small, this behavior can be understood in the proof of Theorem 1 and Proposition 1 to follow.

For the plant given by (6), the closed-loop equation corresponding to Fig. 1 can be written as

$$\dot{x}(t) = Ax(t) + br(t) + \epsilon\Psi(x), \qquad x(0) = x_0;\ x \in \mathbb{R}^n \tag{7}$$

where A is stable and $\Psi(x)$ represents the time-invariant nonlinearity of the system, with $\Psi(0) = 0$. We think of ϵ as being a small and fixed term. Setting $\epsilon = 0$, we obtain the following equation:

$$\dot{x}(t) = Ax(t) + br(t) = f(x, t), \qquad x(0) = x_0;\ x \in \mathbb{R}^n \tag{8}$$

which we refer to as the unperturbed equation. If the periodic reference signal $r(t)$ has a period $T = 2\pi/\omega$, then $f(x, t) = f(x, t + T)$.

Assumption A1): The linear subsystem has no zeros at $\pm jn\omega$, and the realizations are minimal.

Assumption A2): $\Psi(x)$ is as smooth as $f(x)$.

Theorem 1: If assumptions A1) and A2) hold, then for sufficiently small ϵ a) there exists a periodic solution of the perturbed system (7) and b) stability of the unperturbed periodic solution implies that the periodic solution of the perturbed system is also stable.

Proof: Since $r(t)$ has period $2\pi/\omega$, we can rewrite (8) in the form of an autonomous equation in $n+1$ dimensions by defining the function

$$\begin{aligned} \theta&: \quad \mathbb{R}^1 \to \mathbb{S}^1 \\ \theta&: \quad t \mapsto \omega t (\mathrm{mod}\, 2\pi). \end{aligned} \tag{9}$$

Using (9), (8) becomes

$$\begin{aligned} \dot{x}(t) &= f(x, \theta) \\ \dot{\theta}(t) &= \omega; \qquad (x, \theta) \in \mathbb{R}^n \times \mathbb{S}^1. \end{aligned} \tag{10}$$

Similarly, (7) becomes

$$\begin{aligned} \dot{x}(t) &= f(x, \theta) + \epsilon\Psi(x) \\ \dot{\theta}(t) &= \omega; \qquad (x, \theta) \in \mathbb{R}^n \times \mathbb{S}^1. \end{aligned} \tag{11}$$

We define the flow $\phi_t(x_0, \theta_0, \epsilon) := (x(t), \theta(t))$ to be the solution to (11) with the initial conditions x_0 and θ_0. Setting $\epsilon = 0$, we define the flow for the unperturbed system as $\phi_t(x_0, \theta_0, 0) := (x(t), \theta(t)) := (\omega t + \theta_0) \bmod (2\pi)$, where

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)r(s)\,ds. \tag{12}$$

Because the forcing $r(t)$ is periodic, the solution $x(t)$ to the linear system is also periodic in steady state [since A is the stable closed-loop system matrix; see (7)].

Next, consider a Poincaré map P near the periodic flow, $\phi_t(x_0, \theta_0, 0)$. We define a cross section $\Sigma^{\overline{\theta}_0}$ transverse to the unperturbed flow $\phi_t(x_0, \theta_0, 0)$, corresponding to a constant value of θ, as follows:

$$\Sigma^{\overline{\theta}_0} = \{(x, \theta) \in \mathbb{R}^n \times \mathbb{S}^1 | \theta = \overline{\theta}_0 \in (0, 2\pi]\}.$$

Define a projection Π onto the x variable as follows:

$$\begin{aligned} \Pi&: \quad \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n \\ \Pi&: \quad (x, \theta) \mapsto x \end{aligned}$$

and define a Poincaré map P from $\Sigma^{\overline{\theta}_0}$ onto itself as

$$\begin{aligned} P&: \quad \Sigma^{\overline{\theta}_0} \to \Sigma^{\overline{\theta}_0} \\ P&: \quad x \mapsto \Pi[\Phi_{2\pi/\omega}(x, \overline{\theta}_0, 0)]. \end{aligned}$$

The fixed points of P correspond to the periodic orbits of (10). When a small autonomous perturbation $\epsilon\Psi(x)$, as in (11), is added, the persistence of a fixed point of P can be determined by application of the *implicit function theorem.* If x^* is a fixed point of P, then $P(x^*) = \Pi[\Phi_{2\Pi/\omega}(x^*, \overline{\theta}_0, 0)] = x^*$. Let us define another function G with x^* a fixed point as follows:

$$\begin{aligned} G&: \quad \mathbb{R}^n \times \mathbb{S} \times \mathbb{I} \to \mathbb{R}^n \\ G&: \quad (x, \overline{\theta}_0, \epsilon) \mapsto \Pi\left[\Phi_{2\Pi/\omega}\left(x, \overline{\theta}_0, \epsilon\right)\right] - x \end{aligned}$$

where

$$\epsilon \in \mathbb{I} \subset \mathbb{R}; \quad \mathbb{I} = \{\epsilon \in \mathbb{R} | -\epsilon_0 < \epsilon < \epsilon_0\}$$

note

$$G(x^*, \overline{\theta}_0, 0) = \Pi[\Phi_{2\Pi/\omega}(x^*, \overline{\theta}_0, 0)] - x^* = 0. \tag{13}$$

We can assume, without loss of generality, that $(x^*, \overline{\theta}_0) = (0, 0)$. Then, it is evident that $G(0, 0, 0) = 0$. We wish to determine that if a solution of $G(x, 0, \epsilon) = 0$ for small x and ϵ exists. Now, using (12) and (13), we obtain the derivative of G with respect to x evaluated at (0, 0, 0) as $D_xG(0, 0, 0) = e^{At} - I$, which is nonsingular (since A is stable and has no eigenvalue on imaginary axis). Hence, $\det D_xG(0, 0, 0) \neq 0$. Thus, by the implicit function theorem, a function of ϵ, $\overline{x}(\epsilon)$ exists such that $G(\overline{x}(\epsilon), 0, \epsilon) = 0$, for ϵ sufficiently small and contained in $\mathbb{I}$. Therefore, we have $\Pi[\Phi_{2\Pi/\omega}(\overline{x}(\epsilon), 0, \epsilon)] = \overline{x}(\epsilon)$. Thus, the Poincaré map has fixed points for the perturbed system (although it may shift continuously with ϵ, from its previous position).

This proves the existence of a periodic solution for the perturbed system (7).

It can be proven that if a fixed point of a Poincaré map, constructed near a periodic orbit, is asymptotically stable, then the corresponding periodic orbit is also asymptotically stable [21]. Let $x_\epsilon(t)$ and $x_0(t)$ denote solutions to (7) and (8), respectively. If $|x_\epsilon(t_0) - x_0(t_0)| = \mathcal{O}(\epsilon)$ (of order ϵ), then for $|t - t_0| = \mathcal{O}(1)$, $|x_\epsilon(t) - x_0(t)| = \mathcal{O}(\epsilon)$, as long as the perturbation term is smooth [21]. In other words, if the assumption A2) imposed above holds, the solution $x_\epsilon(t)$ to (7) is a continuous function of ϵ. Therefore, it follows that the entries of the matrix $D_x P(\overline{x}(\epsilon))$ vary continuously with ϵ. Moreover, the eigenvalues of a square matrix are a continuous function of its entries. Hence, the eigenvalues of $D_x P(\overline{x}(\epsilon))$ vary continuously with ϵ, as well. Now, for the unperturbed case, we have $D_x P(x^*) = e^{At}$. Since the eigenvalues of A have negative real parts, we can conclude that the eigenvalues associated with $D_x P(\overline{x}(\epsilon))$ are also stable for sufficiently small ϵ. Thus, the fixed point associated with the perturbed flow is stable if that for the unperturbed one is stable, and so will be the corresponding periodic orbit. ■

Proposition 1: If a repetitive controller $\hat{G}_{\rm rc}$ of order N is placed in the closed loop (Fig. 1) given by (7), then the steady-state error signal $e(t)$ cannot have harmonics of order $\leq N$ (we call this property Nth order tracking).

Proof: This proposition can be proven by contradiction. By Theorem 1, the solutions to the closed-loop (7) are periodic. Hence, all the solutions of the closed-loop system are bounded. Suppose the tracking error has a harmonic content of order i (where $i \leq N$). Then, it can be shown that the signal at the output of the repetitive controller grows unbounded with time due to the pole at the same frequency, whereby a contradiction is reached. Hence, the proposition is proven.

Proposition 2: Let the periodic reference trajectory $r(t)$ with period T be in C^1. If a repetitive controller $\hat{G}_{\rm rc}$ of order N is placed in the closed-loop (Fig. 1) given by (7), then the magnitude of the steady-state error signal $e(t)$ is bounded by $(BT)^2/(\pi\omega N)$, where B is a bound based on the reference trajectory.

Proof: Let the ouput signal of the plant (7) be given by the following equation:

$$y(t) = h(x(t)) \tag{14}$$

where h is a continuous function and $x(t)$ is the solution to (7). Therefore, $y(t) \in C^1$. Since both r and y are in C^1, the tracking error $e(t) = r(t) - y(t)$ is in C^1. This implies $\dot{e}(t)$ is in C^0. Also, $\dot{e}(t)$ is bounded on the compact interval ($t \in [0, T]$). Using input-to-state stability property of the system and assuming zero initial conditions for the states, we can write $\|\dot{e}(t)\| \leq K(\|\dot{r}(t)\| + \|r(t)\|) \leq B$, since r and $\dot{r}$ are bounded on $t \in [0, T]$. From the definition of the *total variation* of $e(t)$ on $[0, T]$ $(V(e(t); [0, T]))$ [22], we have

$$V(e(t); [0, T]) \stackrel{\Delta}{=} \sup\{V(e, P)\colon P \text{ is a partition of } [0, T]\} \tag{15}$$

where $P = \{x_0, x_1, \cdots, x_n\}$; $0 < x_0 < x_1 < \cdots$, $x_n = T$ and $V(E, P) \stackrel{\Delta}{=} \Sigma_{j=1}^n |e(x_j) - e(x_{j-1})|$.

Therefore, $V(e, P) \leq B\Sigma_{j=1}^n |x_j - x_{j-1}| \leq BT$. Since the right-hand side of the inequality is independent of P, we have

$$V(e(t); [0, T]) = \sup\{V(e, P)\} \leq BT. \tag{16}$$

From a theorem of Giardina and Chirlian [23] (quoted in the Appendix), we can conclude that the MSE (ϵ_{MS}^2) is bounded by

$$\epsilon_{MS}^2 \leq \frac{(BT)^2}{\pi\omega N}$$

where $\epsilon_{MS}^2 = \|e(t)\|_2^2$. ■

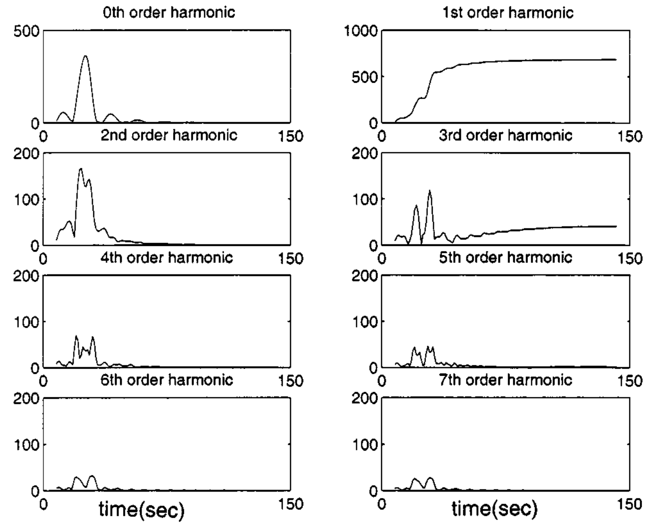

Fig. 8. "Windowed" harmonic-content of the output of the controller in Fig. 1 and given by (7).

We present in Fig. 8 the plots of a harmonic analysis of the output of the controller, which is placed in the closed loop given by (7). The harmonics are obtained by computing the discrete Fourier transform (DFT) over a sliding window of length 100. Since the magnitude of harmonics of order higher than five are seen to be negligibly small in steady state, we can conclude that the chosen model closely approximates the true periodic signal generator. Although Theorem 1 and Proposition 1 were motivated by the ball and beam example, they apply to all plants of the form given by (7). Since a repetitive controller, when applied to a nonlinear plant, can achieve Nth order exact tracking, we can conclude that it is a powerful control methodology for nonlinear as well as linear plants.

This paper was motivated by the noncircular optical turning of eyeglass lenses where actuator nonlinearity limits surface precision. The application of the repetitive controller to optical lathe control is discussed in [24]. The second application of repetitive control that we investigated is the auto-balancing of an active magnetic bearing rotor so that it rotates exactly about its geometric center [25].

V. Conclusion

In this paper, we studied the applicability of the repetitive controller to nonlinear tracking control problems for three different classes of nonlinear systems: 1) with a well-defined relative degree, 2) which fail to have a well-defined relative degree, and 3) linear plants with small actuator nonlinearity. When dealing with the exact linearized model of a nonlinear plant, the use of a repetitive controller in closed loop achieves perfect tracking of a periodic reference signal. The asymptotic convergence of the error can be justified by means of the internal model principle. However, even for the approximate linear models, such as that obtained for the ball and beam system, if the nonlinearity is sufficiently small, the proposed repetitive controller yields better steady state error than that obtained with other methods, such as the feedback controller derived from Jacobian linearization or the feedback controller based on approximate input–output linearized system [18], or approximate backstepping technique when applied to the ball and beam system [19]. Feedback linearization or backstepping are, in general, very powerful nonlinear control methodologies for tracking any trajectory. If the desired trajectory is periodic, however, the use of repetitive control definitely yields much better tracking than other methods.

APPENDIX

Bounds on the Truncation Error [23]: Consider a periodic signal $f(t)$ with period T expressed by the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} \tag{17}$$

where $\omega = 2\pi/T$ and $C_n = 1/T \int_{-T/2}^{T/2} f(t)e^{-jnwt}\,dt$. We approximate $f(t)$ by $f_m(t)$, where

$$f_m(t) = \sum_{n=-m}^{m} C_n e^{jn\omega t}.$$

We wish to determine a value of m, which guarantees the MSE between $f(t)$ and $f_m(t)$ is less than some specified value. The mean square or Hilbert norm is given by

$$\epsilon_{MS}^2 = \|f - f_m\|_2^2 = \int_{-T/2}^{T/2} [f(t) - f_m(t)]^2\,dt.$$

The following theorem gives a bound on the truncation error.

Theorem 2: If a periodic $f(t)$ is of bounded variation whose total variation over one period is bounded by B, that is $V(f, [0, T]) \le B$, then the mean square error is bounded by

$$\epsilon_{MS}^2 \le \frac{B^2}{\pi\omega m}.$$

Proof: See [23].

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Virtual Reference Direct Design Method: An Off-Line Approach to Data-Based Control System Design

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***Abstract*—This paper presents a direct model-reference approach to the off-line design of linear controllers, suited to deal with plants described by a single set of open-loop I/O measurements only. The method is direct inasmuch as the controller parameters are directly estimated with no preliminary identification of any model to describe the plant. The design can be carried out off-line and, in the present formulation, leads to a nonadaptive controller. The basic idea is that of interpreting the open-loop I/O measurements of the plant as closed-loop data produced by a "virtual" reference signal that can be computed by backpropagating the measured output of the plant through the reference model; thus, the controller design reduces to a standard identification problem, in which the "output" signal to be matched is the measured input of the plant. Both a deterministic (noise-free) and a stochastic setting are considered.**

***Index Terms*—Direct control, model reference control, system identification.**

I. INTRODUCTION

The first principle models available for control system design are often simulation models far too complex for standard model-based de-

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