Iterative Learning Control for Nonlinear Nonminimum Phase Plants[1]

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Learning control is a very effective approach for tracking control in processes occurring repetitively over a fixed interval of time. In this paper a robust learning algorithm is proposed for a generic family of nonlinear, nonminimum phase plants with disturbances and initialization error. The ''stable-inversion'' method of Devasia, Chen and Paden is applied to develop a learning controller for linear nonminimum phase plants. This is adapted to accommodate a more general class of nonlinear plants. The bounds on the asymptotic error for the learned input are exhibited via a concise proof. Simulation studies demonstrate that in the absence of input disturbances, perfect tracking of the desired trajectory is achieved for nonlinear nonminimum phase plants. Further, in the presence of random disturbances, the tracking error converges to a neighborhood of zero. A bound on the tracking error is derived which is a continuous function of the bound on the disturbance. It is also observed that perfect tracking of the desired trajectory is achieved if the input disturbance is the same at every iteration.

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1 Introduction

Iterative learning control (ILC) refers to a class of self-tuning controllers where the system performance of a specified task is gradually improved or perfected based on the previous performances of identical tasks. The concept of ''practice'' to improve performance is emphasized in all areas of human endeavor such as gymnastics or a musical performance. In the same way, learning controllers attempt to tune the performance of a system on a given trajectory by learning through practice. Additionally, learning control can be used effectively when the plant cannot be modeled accurately. The most commonly seen applications of learning control are in the area of robot control in manufacturing and production industries. Typically, a robot is required to perform a single task, say pick-and-place an object along a given trajectory, repetitively. With a feedback controller alone, the same tracking error would persist in every repeated trial. In contrast, a learning controller can use the information from the previous execution to improve the tracking performance in the next execution. While in some applications, the need to repeat a trajectory multiple times for learning may be a distinct disadvantage, in many other applications, repetitive tasks are commonly performed making learning control a very natural solution.

For more than a decade researchers have defined and analyzed iterative learning control (ILC) schemes. First introduced in 1984 by Arimoto et al. [1] and Craig [2] and later modified by many others including Kawamura et al. [3], Atkeson et al. [4] and Bondi et al. [5], ILC schemes strive to improve the performance of repetitive tasks using the information of the previous trial of the same task. Modifications of the basic ILC algorithm, such as P-type, PD-type, and PID-type, have evolved in the process. The robustness of the ILC algorithms to disturbances, uncertainties and initialization errors is still an active area of research. Arimoto [6] analyzed the robustness of a PI-type ILC algorithm to errors in initialization, measurement and fluctuation during operation, and introduced a ''forgetting factor'' into the ILC to enhance it. In model-based ILC schemes [4], the inputs corresponding to the desired and actual trajectories are computed from the estimated system parameters and the resulting input error is fed into the learning operator. The performance of this algorithm depends on the quality of the parameter estimates. A more common approach is to operate on the output errors directly. It is shown in [7] that the model-based learning scheme in [4] is a special case of this more general approach. Sugie and Ono [8] demonstrate the necessity of the use of the error derivative in the learning control process for dynamical plants that do not have a direct transmission term. Based on the time-varying nonlinear extension result of Hauser [7], Heinzinger et al. [9] showed that under certain assumptions, the asymptotic tracking errors are bounded and the bounds are continuous functions of the initial errors, uncertainties, and disturbances. Chen et al. [10] proposed a way to adjust the final tracking error bound to a prescribed level with the use of the current iteration tracking error, in the presence of uncertainties, disturbances, and initialization errors.

Gao and Chen [11] have illustrated with counter examples the limitations of some of these learning algorithms with regard to nonminimum phase systems. To remove the minimum phase requirement, they developed a new learning algorithm for linear systems based on ''stable inversion'' [12].

Similar concept of system inversion of nonminimum phase plants appears in the work of Tomizuka [13,14]. Tomizuka et al. describe the noncausal (pseudo-) inversion of plants with known delay for linear discrete-time systems [13,14]. The work of Devasia et al. [12] involves noncausal exact inversion of nonlinear continuous-time systems: both papers use the concepts of acausal filters and the notion of zero-phase appears. In our paper, the controller is acausal and the controller-plant cascade is unity for a linear plant and therefore zero phase at all frequencies. The controller-plant cascade in [13,14] is zero-phase at all frequencies but unity gain only at low frequencies.

Learning control takes advantage of previous attempts at executing a particular trajectory to do better in successive attempts. Such a system is depicted in Fig. 1 where a desired trajectory y_d is to be tracked (as closely as possible). An initial input u_0 drives the system, the corresponding output y_0 is compared to y_d and the error is used to compute the next trial input u_1. An underlying

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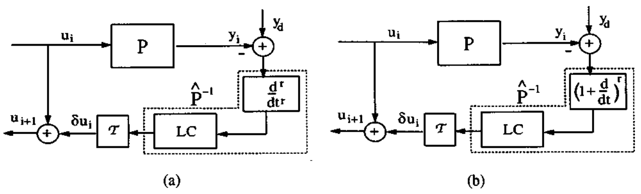

Fig. 1 Iterative learning control for linear plants. (*a*) Update law is given by (9); (*b*) Update law is given by (10). *LC*: learning controller, $\mathcal{T}$: truncation operator.

assumption or constraint in learning control is that the system is reinitialized at each new trial. This may be accomplished via an additional control or it may be an approximation (e.g., if the system is stable and a zero input is applied, the system state is pretty close to 0 after several time constants). This contrasts with the formulation in repetitive control where no such resetting occurs.

The important fact presently known about the learning control paradigm is that it is a powerful method of achieving high-performance tracking in practical systems. Further, convergence results for the methods referenced above give us some understanding that the systems for which learning control is applicable is still limited. For example, it was not known how to design learning control algorithms for systems with relative degree greater than one, and for nonlinear systems requiring globally convergent algorithms. Also, in this work, like many other previous work it is necessary to differentiate the output which restricts the class of nonlinear plants, for example: nonlinear plants with output sensor noise cannot be considered. In this paper, we devise a scheme that eliminates the relative-degree one constraint. Our results are local however. In addition, through the use of noncausal inversion of the plant we state our assumptions in relatively straightforward terms. In contrast to most other ILC algorithms, which use abstract constraints on the plant and controller without clear physical interpretations, the establishment of convergence in our algorithm is based on relatively straightforward assumptions with unambiguous physical meanings.

Finally, the inversion-based approach we take in this paper converges in one iteration in the linear case. Thus we expect that our scheme will have rapid convergence when applied to nonlinear systems meeting the assumptions of smoothness that we impose. We establish convergence but not convergence rates in the nonlinear case however. In this paper, we propose a robust iterative learning control algorithm for a class of nonlinear nonminimum phase plants with input disturbances. In Section 2, a ''stable inversion'' based noncausal learning algorithm is derived for linear nonminimum phase plants. A modification of this algorithm is also proposed to make it more robust. In Section 3, we extend this stable inversion based ILC scheme to nonlinear nonminimum phase plants and input disturbances. A proof of convergence of the input trajectory to a neighborhood of the desired one is also provided. Simulation examples are presented to show the performance of the proposed learning controller. In Section 4, it is observed that near perfect tracking of the desired trajectory is achieved if the input disturbance is the same at every iteration of the ILC. Finally, Section 5 concludes the paper.

2 Stable Inversion Based Learning Algorithm

In this section we briefly review the stable inversion based learning algorithm (see [12,11]) and propose a modification of the same to improve robustness. Consider a plant P, assumed to be linear, for simplicity. We also assume: (1) the plant (P) is SISO; (2) the relative degree r of the plant is known; (3) the plant is asymptotically stable; (4) P has no zeros on the $j\omega$ axis; (5) the desired trajectory $y_d(t)$, $t\in[0,T]$ is smooth, specifically $y_d(\cdot)\in C^r[0,T]$; (6) output disturbances are absent.

2.1 Learning Control Algorithm for Linear Plants.

For a system satisfying the above assumptions, a learning control scheme is described as shown in Fig. 1(*a*). The learning operator is thought of as an inverse of the given plant. If the inverse system is stable, the update law of the learning controller is obtained as $u_{i+1}(t)=u_i(t)+P^{-1}[y_d(t)-y_i(t)]$. The update law (with $u_0=\mathcal{T}(u_0)$) is given as:

$$\begin{aligned}u_{i+1}(t)&=\mathcal{T}(u_i(t)+P^{-1}[y_d(t)-y_i(t)]);\\&=u_i(t)+\mathcal{T}(u_i(t)+P^{-1}[y_d(t)-y_i(t)]);\\&=u_i(t)-u_i(t)+\mathcal{T}(P^{-1}y_d(t))\\&=\mathcal{T}(P^{-1}y_d(t)),\end{aligned}\tag{1}$$

where $\mathcal{T}$ is the truncation operator, such that $\mathcal{T}(x(t))=x(t)\forall t\in[0,T]$; and $\mathcal{T}(x(t))=0$ otherwise and P^{-1} is the mapping that give us the input u corresponding to an output i.e., P^{-1}: $L_\infty\to L_\infty$; P^{-1}: $y_d(\cdot)\mapsto u_i(\cdot)$. It is to be noted that if $u_0=\mathcal{T}(u_0)$, $\mathcal{T}(u_i(t)+\delta u_i(t))=u_i(t)+\mathcal{T}(\delta u_i(t))$ for all i. (Note that in Fig. 1 the truncation operator $\mathcal{T}$ is placed before the summing junction.) If the system has relative degree $r=1$, then the plant can be written in state space form as:

$$\begin{gathered}\dot{x}(t)=Ax(t)+Bu(t),\quad x(0)=0,\\y(t)=Cx(t)\\\dot{y}(t)=C\dot{x}(t)=CAx(t)+CBu(t);\ \text{where}\ CB\neq0;\end{gathered}\tag{2}$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times1}$, $C\in\mathbb{R}^{1\times n}$, $D\in\mathbb{R}$; $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}$. Since the plant is stable, we can replace $x(0)=0$ with $x(\pm\infty)=0$ and not alter the I-O map defined by (2). If the relative degree $r>1$ $(CB=0)$ check the second derivative:

$$\ddot{y}(t)=CA^2x(t)+CABu(t)\ \text{ If }\ r=2,\ CAB\neq0,\ \text{else}\ CAB=0\tag{3}$$

Then continue differentiating $y(t)$ until $u(t)$ appears algebraically for the first time:

$$y^r(t)=CA^rx(t)+CA^{r-1}Bu(t)\quad\text{and}\quad CA^{r-1}B\neq0,\tag{4}$$

where, $y^r(t)$ denotes the rth derivative of $y(t)$. Similarly, we can write:

$$y_d^r(t)=CA^rx(t)+CA^{r-1}Bu_d(t)\tag{5}$$

Thus, solving (4) for u leads to:

$$u(t)=(CA^{r-1}B)^{-1}[y^r(t)-CA^rx(t)]\tag{6}$$

Therefore, $\delta u(t)=u_d(t)-u(t)=-(CA^{r-1}B)^{-1}CA^rx(t)+(CA^{r-1}B)^{-1}[y_d^r(t)-y^r(t)]$. Substituting $\delta u(t)$ in (2) in place of $u(t)$ gives:

$$\begin{gathered}\dot{\overline{x}}_i(t)=\overline{A}\overline{x}_i(t)+\overline{B}(y_d^r(t)-y_i^r(t)),\ \ \overline{x}_i(\pm\infty)=0\\\delta u_{i+1}(t)=\overline{C}\overline{x}_i(t)+\overline{D}(y_d^r(t)-y_i^r(t))\end{gathered}\tag{7}$$

where i is the index of iteration and

$$\begin{gathered}\overline{A}=A-B(CA^{r-1}B)^{-1}CA^r\in\mathbb{R}^{n\times n}\\\overline{B}=B(CA^{r-1}B)^{-1}\in\mathbb{R}^{n\times1}\\\overline{C}=-(CA^{r-1}B)^{-1}CA^r\in\mathbb{R}^{1\times n}\\\overline{D}=(CA^{r-1}B)^{-1}\in\mathbb{R}\end{gathered}$$

Since the linear controller described by the above equations may have unstable eigenvalues, the noncausal solution approach of [12] is adopted under the boundary condition assumption $\overline{x}_i(\pm\infty)=0$ (see Appendix D). Since the operation $P^{-1}(y_d(t)-y(t))$ (in (1)) is first performed and then the solution is truncated we take the boundary values of P^{-1} at $\pm\infty$. For a linear system this simply means that the impulse response decays away exponentially in both time directions. This is the time-domain framework consistent with the use of Fourier transforms. Closed-form solution exists for linear disturbance-free case. For a real matrix A (in (7)), there exists an invertible $n \times n$ matrix V such that $J = V^{-1}AV$, where J is the real Jordan form of A. Therefore with the coordinate transformation $x = Vz$ and decoupling the system into "stable" and unstable subsystems, we can rewrite the controller as:

$$\begin{bmatrix} \dot{z}^s(t) \\ \dot{z}^u(t) \end{bmatrix} = \begin{bmatrix} A^s & 0 \\ 0 & A^u \end{bmatrix} \underbrace{\begin{bmatrix} z^s(t) \\ z^u(t) \end{bmatrix}}_{Z} + \begin{bmatrix} B^s \\ B^u \end{bmatrix} (y_d^r(t) - y^r(t)), \quad \begin{bmatrix} z^s(\pm\infty) \\ z^u(\pm\infty) \end{bmatrix} = 0$$

$$\delta u_{i+1}(t) = [C^s \; C^u] \begin{bmatrix} z^s(t) \\ z^u(t) \end{bmatrix} + D(y_d^r(t) - y^r(t)) \tag{8}$$

where A^s has all its eigenvalues in the closed left-half plane (it will have r zero eigenvalues) and A^u has all its eigenvalues in the open right-half of the complex plane. The system can thus be decoupled into a "stable subsystem" and an unstable subsystem. With the boundary conditions $Z(\pm\infty) = 0$, the stable subsystem is integrated forward in time to calculate $z^s(t)$, while the unstable subsystem is integrated backward in time to calculate $z^u(t)$. However, a limitation of this algorithm is that, the calculations of $z^s(t)$ is difficult, since A^s has r zero eigenvalues. The learning controller is not robust to initialization errors and noise. To alleviate this problem we propose a modification based on the generalization of the following example. Let a linear plant with relative degree $r = 1$ be represented by the following transfer function.

$$Y(s) = H(s)U(s); \quad \text{therefore,}$$

$$\dot{Y}(s) \triangleq \mathcal{L}\left(\frac{d}{dt}y(t)\right) = sH(s)U(s) - y(0);$$

where $\mathcal{L}$ represents Laplace transform, $y(0)$ is the initial condition on y. This implies that the transfer function of the learning controller is:

$$\delta U(s) \triangleq U_d(s) - U(s) = \underbrace{\frac{1}{s}H^{-1}(s)}_{\textit{learning controller}} [\dot{Y}_d(s) - \dot{Y}(s)], \tag{9}$$

neglecting the initialization error. Note that the transfer function of the learning controller has a pole at $s = 0$. However, taking $\delta y + \delta\dot{y}$ as the input to the learning controller, rather than $\delta\dot{y}$, it is seen that the controller transfer function has a stable pole at $s = -1$ *instead of at* $s = 0$ *as shown below*

$$\begin{aligned} \delta U(s) &= \underbrace{\frac{1}{(s+1)}H^{-1}(s)}_{\textit{learning controller}} [\dot{Y}_d(s) + Y_d(s) - \dot{Y}(s) - Y(s)]; \\ &= \underbrace{\frac{1}{(s+1)}H^{-1}(s)(s+1)}_{P^{-1}} [Y_d(s) - Y(s)]; \end{aligned} \tag{10}$$

In time domain the learning controller can be split into stable and unstable parts as shown in Eq. (8) to obtain a stable noncausal inverse of the plant. It can be seen that the modified learning controller is more robust to initialization errors. If the relative degree $r > 1$, the input to the learning operator is given by: $\mathcal{P}r(d/dt)e$ where $\epsilon = y_d - y_i$ and $\mathcal{P}r$ is a stable polynomial of order r as shown in Fig. 1(b).

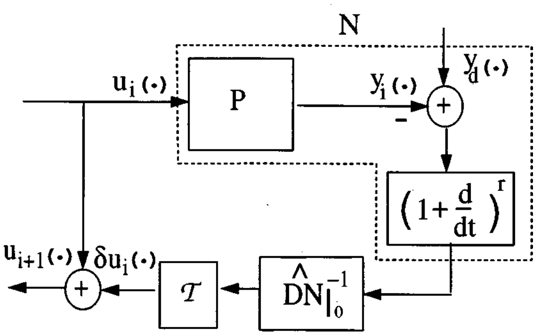

Fig. 2 Nonlinear learning control system. $DN|_0^{-1}$ **is the learning controller.**

3 Nonlinear Nonminimum Phase Plant With Input Disturbances

In this section, we present a robust iterative learning algorithm for nonlinear systems with well-defined relative degree. We consider only time-invariant nonlinear systems in this section.

3.1 System Description. Consider a nonlinear system which is *stable-in-first-approximation* at $x = 0$ and also input-to-state stable:

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + g(x_i(t))u_i(t), \quad x_i(0) = 0 \\ y_i(t) &= h(x_i(t)); \end{aligned} \tag{11}$$

where i is the index of iteration of ILC, $\{u_i\}_{i=0}^{\infty}$ is the input sequence, $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $y_i(t) \in \mathbb{R}^m$ and $f: \mathbb{R}^n \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $h: \mathbb{R}^n \to \mathbb{R}^m$. For the sake of simplicity we consider the case relative degree $r = 1$. The objective of learning is to construct a sequence of input trajectories $\{u_i\}_{i=1}^{\infty}$ such that $u_i \to u^*$ and $u^*(t)$ causes the system to track a trajectory $y_d(t)$ "as closely as possible" on $[0,T]$

$$\begin{aligned} \dot{x}_d(t) &= f(x_d(t)) + g(x_d(t))u_d(t), \quad x_d(0) = 0, \\ y_d(t) &= h(x_d(t)); \end{aligned} \tag{12}$$

is satisfied $\forall t \in [0,T]$. To model input disturbances, the plant equation (11) is modified:

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + g(x_i(t))u_i(t) + b(x_i(t))w_i(t), \quad x_i(0) = 0 \\ y_i(t) &= h(x_i(t)); \end{aligned} \tag{13}$$

where $b: \mathbb{R}^n \to \mathbb{R}^n$ and $w_i(t) \in \mathbb{R}$. The function $w_i(t)$ represents both deterministic and random bounded disturbances of the system; it may be stiction, nonreproducible friction, modeling errors, etc.

We make the following assumptions:

(A1) The functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, $h_x(\cdot)$ are continuously differentiable and $b(\cdot)$ is continuous.

(A2) $y_d(\cdot) \in C^r$, where $r = 1$ is the relative degree of the system.

(A3) $u_0 \in L_\infty \cap C^0 \cap B_r$.

(A4) The system is *stable-in-first-approximation* and input-to-state stable. (Note: If the system is not stable, it may be stabilized prior to application of our methods.)

(A5) The system has hyperbolic zero-dynamics.

(A6) The disturbance $w_i(\cdot)$ is bounded by b_w (i.e., $\|w_i(t)\| \leq b_w$) and continuous.

For such a system a learning algorithm is proposed as shown in Fig. 2. This learning control scheme is very similar to Fig. 1(b), except that P in Fig. 1(b) is a linear plant while P here refers to a nonlinear plant.

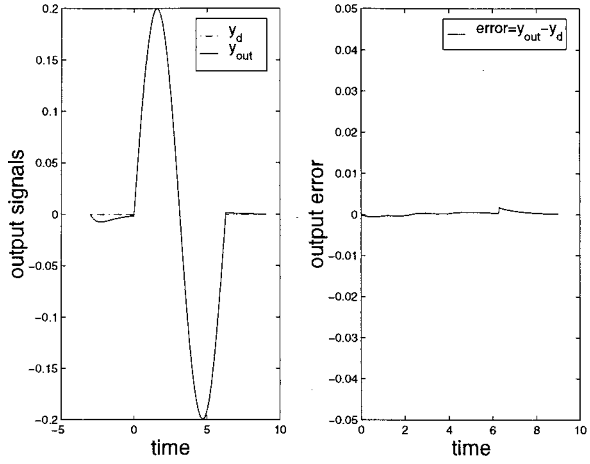

Fig. 3 Tracking of nonlinear nonminimum phase system without input disturbance after 3 iterations

3.2 Formulation of the Learning Controller. In this section, we derive a good candidate for the learning controller $DN|_0^{-1}$ (shown in Fig. 2) by linearizing the plant. From the stable nonlinear system (13) we define an input-to-output nonlinear plant P (stable in first approximation) as follows:

$$P : u_i \mapsto y_i .$$

If $x_i(t)$ is a solution to the differential equation (13), then $\|x_i\|_\infty < M < \infty$ (by the input-to-state stability assumption). From the compactness of x_i and continuity of the function h and using Theorem 4.15 in [15] we can write from (13): $y_i(\cdot) \in L_\infty$. By the chain rule $\dot{y}_i(t) = h_x(x_i(t))\dot{x}_i(t)$. Again from the differentiability of x_i, h and h_x (Assumption A1), and using Theorem 5.5 in [15] we obtain: $y_i(\cdot) \in C^1$. Hence $\dot{y}_i(\cdot) \in C^0$. From Theorem 4.15 in [15] we obtain: $\dot{y}_i(\cdot) \in L_\infty$. Hence we have.

$$P : u_i \mapsto y_i \tag{14}$$

$$P : C^0 \cap L_\infty \to C^1 \cap L_\infty \tag{15}$$

Define a linear operator $DT \triangleq (1 + d/dt)$. Thus,

$$DT(\delta y_i) = \delta y_i + \delta \dot{y}_i \tag{16}$$

where, $\delta y_i = y_d - y_i$ and $\delta \dot{y}_i = \dot{y}_d - \dot{y}_i$. We have assumed $y_d \in C^r$ ($r = 1$ is the relative degree) and $y_d \in L_\infty$ where y_d is the desired output-trajectory. Also we have proved that $y_i \in C^1 \cap L_\infty$. Hence, $\delta y = y_d - y_i \in C^1 \cap L_\infty$. By similar arguments we can say, $\delta \dot{y} = \dot{y}_d - \dot{y}_i \in C^0 \cap L_\infty$. Therefore,

$$DT : \delta y_i \mapsto \delta y_i + \delta \dot{y}_i$$

$$DT : C^1 \cap L_\infty \to C^0 \cap L_\infty .$$

Define a nonlinear operator N as:

$$\begin{aligned} N &\triangleq -(DT \circ P) : u_i \mapsto -(\delta y_i + \delta \dot{y}_i) \\ &C^0 \cap L_\infty \to C^0 \cap L_\infty . \end{aligned} \tag{17}$$

A linear operator $DP|_0$ is defined by linearizing the system (13) around ($x = 0$, $u = 0$, $w = 0$) as follows:

$$\begin{aligned} \delta \dot{\overline{x}}_i(t) &= \overline{A}\,\delta \overline{x}_i(t) + \overline{b}\,\delta \overline{u}_i(t), \quad \delta \overline{x}_i(0) = 0 \\ \delta \overline{y}_i(t) &= \overline{C}\,\delta \overline{x}_i(t). \end{aligned} \tag{18}$$

where $\overline{A} \triangleq f_x(0)$; $\overline{b} \triangleq g(0)$; $\overline{C} \triangleq h_x(0)$.

Since the plant is stable in first approximation, $\overline{A}$ is Hurwitz in (18) we can replace $\delta \overline{x}_i(0) = 0$ with $\delta \overline{x}_i(\pm\infty) = 0$ and not alter the I-O map defined by (18).

Defining $DN|_0$ & $DN|_0^{-1}$

Now we define a linear operator $DN|_0$ as:

$$DN|_0 \triangleq DT \circ DP|_0 : \delta \overline{u}_i \mapsto \delta \overline{y}_i + \delta \dot{\overline{y}}_i \tag{19}$$

Thus, $DN|_0(\delta \overline{u}_i) = (\overline{C} + \overline{CA})\,\delta \overline{x}_i(t) + \overline{Cb}\,\delta \overline{u}_i(t) = \delta \overline{y}_i + \delta \dot{\overline{y}}_i$, where, $\delta \overline{x}_i$ is a solution to (18)

$$(\delta \dot{\overline{y}}_i \in C^0 \cap L_\infty).$$

Therefore,

$$DN|_0 : C^0 \cap L_\infty \to C^0 \cap L_\infty .$$

The linear operator $DN|_0$ is invertible, since $r = 1$, $\overline{Cb} \neq 0$ (see proof of Lemma 1). The learning controller ($DN|_0^{-1}$) is defined by the following equations:

$$\begin{aligned} \delta \dot{\overline{x}}_i(t) &= \overline{A}\,\delta \overline{x}_i(t) + \overline{b}(\overline{Cb})^{-1}[\delta \overline{y}_i(t) + \delta \dot{\overline{y}}_i(t) \\ &\quad - (\overline{C} + \overline{CA})\,\delta \overline{x}_i(t)], \quad \overline{x}_i(\pm\infty) = 0 \\ &= [\overline{A} - \overline{b}(\overline{Cb})^{-1}(\overline{C} + \overline{CA})]\,\delta \overline{x}_i(t) \\ &\quad + \overline{b}(\overline{Cb})^{-1}[\delta \overline{y}_i(t) + \delta \dot{\overline{y}}_i(t)]; \\ \delta \overline{u}_i(t) &= (\overline{Cb})^{-1}[\delta \overline{y}_i(t) + \delta \dot{\overline{y}}_i(t) - (\overline{C} + \overline{CA})\,\delta \overline{x}_i(t)]. \end{aligned} \tag{20}$$

Since the system is hyperbolic (Assumption 5), with the boundary conditions we can obtain the solution for the above system using the ‘‘stable noncausal’’ approach of [12]. Hence the linear operator $DN|_0^{-1}$ is defined as:

$$DN|_0^{-1} : C^0 \cap L_\infty \to C^0 \cap L_\infty ;$$

$$(\delta \overline{y}_i + \delta \dot{\overline{y}}_i) \mapsto \delta \overline{u}_i ;$$

Denoting $\delta x_i = x_d - x_i$, $\delta y_i = y_d - y_i$ and $\delta u_i = u_d - u_i$, we derive (by Taylor series expansion) a linearized plant from (13) as follows:

$$\begin{aligned} \dot{x}_i(t) + \delta \dot{x}_i(t) &= f(x_i(t) + \delta x_i(t)) + g(x_i(t) + \delta x_i(t)) \cdot [u_i(t) \\ &\quad + \delta u_i(t)]; \\ &\approx f(x_i(t)) + f_x(x_i(t))\,\delta x_i(t) + [g(x_i(t)) \\ &\quad + g_x(x_i(t))\,\delta x_i(t)][u_i(t) + \delta u_i(t)]; \end{aligned} \tag{21}$$

(by Taylor series expansion and neglecting h.o.t) where

$$f_x(x_i(t)) \triangleq \frac{\partial f}{\partial x}(x_i(t)); \quad g_x(x_i(t)) \triangleq \frac{\partial g}{\partial x}(x_i(t))$$

Also, $y_i(t) + \delta y_i(t) = h(x_i(t) + \delta x_i(t))$.

Subtracting (13) from (21) and neglecting h.o.t we obtain the linearized plant (around the solution $x_i(t)$ to (13) as:

$$\begin{aligned} \delta \dot{x}_i(t) &= f_x(x_i(t))\,\delta x_i(t) + g(x_i(t))\,\delta u_i(t) + g_x(x_i(t))\,\delta x_i(t) u_i(t) \\ &\quad - b(x_i(t)) w_i(t), \quad \delta x_i(0) = 0 \\ \delta y_i(t) &= h_x(x_i(t))\,\delta x_i(t); \end{aligned} \tag{22}$$

Since (18) is stable, it can be proved by Lyapunov methods that (22) is also BIBO stable if x_i lies within a certain bound. Note that, here also we can replace $\delta x_i(0) = 0$ (as in 18) with $\delta x_i(\pm\infty) = 0$ and not alter the I-O map. Define $A_i(t) \triangleq f_x(x_i(t)) + g_x(x_i(t)) u_i(t)$, $B_i(t) \triangleq g(x_i(t))$, $C_i(t) \triangleq h_x(x_i(t))$, $b_i(t) \triangleq b(x_i(t))$. The stable linear system (22) has a solution and defines a linear I-O map:

$$DP|_{u_i} : \delta u_i \to \delta y_i ; \quad C^0 \cap L_\infty \to C^1 \cap L_\infty .$$

$\delta \dot{y}(t) = h_x(x_i(t))\,\delta \dot{x}_i(t) + h_{xx}(x_i(t)) \dot{x}_i \delta x_i(t)$. We can argue, $\delta \dot{y}(\cdot) \in C^0 \cap L_\infty$. Define a linear operator $DN|_{u_i}$ as:

$$\begin{gathered} DN|_{u_i} \triangleq DT \circ DP|_{u_i} : \delta u_i \mapsto \delta y_i + \delta \dot{y}_i \\ C^0 \cap L_\infty \rightarrow C^0 \cap L_\infty \end{gathered} \tag{23}$$

In an iterative learning control process, as shown in Fig. 2, at each learning step *i*, the control input u_i is updated as $u_{i+1} = \mathcal{T}(u_i + \delta\overline{u}_i)$. The sum of the error in the output signal ($\delta\overline{y}_i \triangleq y_d - y_i$) and the error in its derivative ($\delta\dot{\overline{y}}_i \triangleq \dot{y}_d - \dot{y}_i$) is the input to the learning operator, while $\delta\overline{u}_i$ is the output. Since it is assumed in this section that the nonlinear system is of relative degree 1, it is necessary to take the derivative of the output to invert the system. In practice, the differentiation can only be approximated because of the presence of the output sensor noise. For the clarification of the remaining discussion, function parameters will be shown in subscript notation with the dependence of time implied unless otherwise stated.

Now the iterative update law of ILC can be written in terms of the operator *N* and $DN|_0^{-1}$ as follows:

$$u_{i+1} = \mathcal{T}(u_i + \delta\overline{u}_i) = \mathcal{T}(u_i + (\overline{Cb})^{-1}[\delta\overline{y}_i + \delta\dot{\overline{y}}_i - (\overline{C} + \overline{CA})\delta\overline{x}_i]),$$

where, $\delta\overline{x}_i$ is a solution to (18)

$$= u_i + \mathcal{T}(DN|_0^{-1}(\delta\overline{y}_i + \delta\dot{\overline{y}}_i)) = u_i - \mathcal{T}(DN|_0^{-1}N(u_i)) \tag{24}$$

We now present the following lemma establishing when a noncausal solution is required for a nonlinear plant:

Lemma 1. If a nonlinear system with a well-defined relative degree and hyperbolic zero-dynamics is nonminimum-phase (i.e., zero-dynamics is unstable), the linearized plant (about equilibrium point (0,0)) has unstable zeros.

Proof: See Appendix A.

From Lemma 1 we can conclude that if the given nonlinear system has unstable zero-dynamics, then $DN|_0^{-1}$ has unstable eigenvalues, and hence it is necessary to apply the stable noncausal inversion process [12] on $DN|_0$ as described in the previous section.

3.3 Convergence Analysis. *Definition 1. We define the* λ *norm of a function* $x:[0,T] \rightarrow \mathbb{R}^k$ by

$$\|x(\cdot)\|_\lambda \triangleq \sup_{t \in [0,T]} e^{-\lambda t} \|x(t)\| \tag{25}$$

Note that $\|x\|_\lambda \leqslant \|x\|_\infty \leqslant e^{\lambda T}\|x\|_\lambda$ for $\lambda > 0$, implying that these two norms are equivalent. Thus convergence results can be proved using either norm.

Theorem 1. If the assumptions (A1–A6) imposed above hold, then the algorithm (24) produces a sequence of inputs which converge to u^* if there are no input disturbances (i.e., $w_i = 0$) and no initialization error. If w_i is bounded, u_i converges to $B(u^*,r)$ as $i \rightarrow \infty$. The radius r of the ball $B(u^*,r)$ depends continuously on the bound on the disturbance w_i. If there exists a $u_d \in L_\infty \cap C_0[0,T]$ with $P(u_d) = y_d$, then u_i converges to the desired input solution u_d.

The proof relies on the application of a variant of the contraction mapping theorem [16] to the input sequence. The main idea is to show that $\|\delta\overline{u}_{i+1}\|_\lambda \leqslant \rho \|\delta\overline{u}_i\|_\lambda + b_d$, where $0 \leqslant \rho < 1$ $(\delta\overline{u}_i \triangleq u_d - u_i)$. This implies that $\lim_{i \rightarrow \infty} \sup \|\delta\overline{u}_i\|_\lambda \leqslant (1-\rho)^{-1} b_d$, where b_d is a continuous function of the bound on disturbance.

Road Map to Proof: First a mapping *T* is defined from u_i to u_{i+1}. Using the fact that $DN|_{u_i}$ is the Frechet derivative of *N* and bounds on the functions like *f*, *g* and h_x it is proved that *T* is a contraction mapping. Hence the input sequence $\{u_i(\cdot)\}_{i=0}^\infty$ converges to a ball $B(u^*,r)$. It is also proved that in the absence of disturbances and initialization error u_i converges to u^*. Moreover, if $\exists u_d$ such that $P(u_d) = y_d$, the fixed point u^* of the contraction mapping $T_i(\cdot)$ is shown to be u_d.

Proof: Construct the sequence $\{u_i(\cdot)\}_{i=0}^\infty$ by defining:

$$u_0 = \mathcal{T}(u_0),$$

(i.e., u_0 has compact support)

$$u_{i+1} = T_{x_i(\cdot)}[u_i(\cdot)] \triangleq \mathcal{T}(u_i - DN|_0^{-1}N(u_i)) \tag{26}$$

where $T_{x_i(\cdot)}[u_i(\cdot)]$ is denoted by $T_i(u_i)$ for simplicity in the rest of this paper. Now from (24) we can write

$$\begin{aligned} \|T_i(u_i) - T_i(v_i)\| &= \|\mathcal{T}(u_i - DN|_0^{-1}N(u_i) - v_i + DN|_0^{-1}N(v_i))\| \\ &= \|\mathcal{T}(u_i - DN|_0^{-1}N(u_i) - v_i \\ &\quad + DN|_0^{-1}N(u_i + v_i - u_i))\| \end{aligned} \tag{27}$$

Paden and Chen in [17] show that the Frechet derivative of *N* is given by $DN|_{u_i}$ (see Appendix C, Proposition 1). That is:

$$\lim_{|\delta u_i| \rightarrow 0} \frac{\|N(u_i + \delta u_i) - N(u_i) - DN|_{u_i}[\delta u_i]\|}{\|\delta u_i\|} = 0 \tag{28}$$

Let $s(\delta u_i)$ be defined as:

$$s(\delta u_i) \triangleq N(u_i + \delta u_i) - N(u_i) - DN|_{u_i}[\delta u_i].$$

From (28) we can see *s* is $o(\delta u_i)$. Denoting $\delta u_i \triangleq [v_i - u_i]$ we can rewrite (27) as:

$$\begin{aligned} \|T_i(u_i) - T_i(v_i)\| = \|\mathcal{T}(u_i - DN|_0^{-1}N(u_i) - v_i + DN|_0^{-1}N(u_i) \\ + DN|_0^{-1}DN|_{u_i}[\delta u_i] + DN|_0^{-1}s(\delta u_i))\| \end{aligned} \tag{29}$$

Again from the definition of the operator $DN|_{u_i}$ (23) we obtain:

$$DN|_{u_i} : \delta u_i \mapsto \delta\overline{y}_i + \delta\dot{\overline{y}}_i$$

$$\begin{aligned} \delta\overline{y}_i + \delta\dot{\overline{y}}_i = [&h_x(x_i) + h_x(x_i)f_x(x_i) + h_x(x_i)g_x(x_i)u_i + h_{xx}(x_i)f(x_i) \\ &+ h_{xx}(x_i)g(x_i)u_i + h_{xx}(x_i)b(x_i)w_i]\delta x_i \\ &+ h_x(x_i)g(x_i)\delta u_i - h_x(x_i)b(x_i)w_i; \end{aligned}$$

From the definition of the linear operator $DN|_0^{-1}$ (20), $h_x(0) = \overline{C}$, $g(0) = \overline{b}$ (18) we can write:

$$DN|_0^{-1} : \delta\overline{y}_i + \delta\dot{\overline{y}}_i \mapsto \delta\overline{u}_i;$$

$$\begin{aligned} DN|_0^{-1}DN|_{u_i}\delta u_i = \delta\overline{u}_i = (h_x(0)g(0))^{-1}[&(h_x(x_i) + h_x(x_i)f_x(x_i) \\ &+ h_x(x_i)g_x(x_i)u_i + h_{xx}(x_i)f(x_i) \\ &+ h_{xx}(x_i)g(x_i)u_i + h_{xx}(x_i)b(x_i)w_i)\delta x_i \\ &+ h_x(x_i)g(x_i)\delta u_i - h_x(x_i)b(x_i)w_i - (h_x(0) \\ &+ h_x(0)f_x(0))\delta\overline{x}_i]; \end{aligned}$$

Since $s(\delta u_i)$ is $o(\delta u_i)$,

$$\lim_{|\delta u_i| \rightarrow 0} \frac{\|DN|_0^{-1}\| \|s(\delta u_i)\|}{\|\delta u_i\|} = 0. \tag{30}$$

This implies that $\forall \epsilon_1 > 0$, $\exists \delta_1 > 0$ such that $\|\delta u_i\| \leqslant \delta_1$ implies $|DN|_0^{-1}|\|s(\delta u_i)\|/\|\delta u_i\| \leqslant \epsilon_1$.

Showing T is a Contraction Mapping

Since x_i belongs to a compact set, $b(x_i)$, $f(x_i)$, $g(x_i)$, $h_x(x_i)$, $f_x(x_i)$, $g_x(x_i)$ and $h_{xx}(x_i)$ are bounded and let b_b, b_f, b_g, b_{hx}, b_{fx}, b_{gx}, b_{hxx} be the respective norm bounds. Denoting $l = (\overline{Cb})^{-1}(h_x(x_i)g(x_i)) \approx 1$ and $\epsilon = |1 - l| < 1$, choosing an ϵ_1 such that $(\epsilon + \epsilon_1) < 1$ and using (30), (29) can be rewritten as:

$$
\begin{aligned}
\|T_i(u_i)-T_i(v_i)\| &= \|\mathcal{T}(u_i-v_i+DN|_0^{-1}DN|_{u_i}[\delta u_i]+DN|_0^{-1}s(\delta u_i))\| \\
&\leqslant \|\mathcal{T}(u_i-v_i+DN|_0^{-1}DN|_{u_i}[\delta u_i])\|+DN|_0^{-1}s(\delta u_i)\| \\
&\leqslant \|\mathcal{T}(-\delta u_i+(\overline{Cb})^{-1}[(h_x(x_i)+h_x(x_i)f_x(x_i)+h_x(x_i)g_x(x_i)u_i+h_{xx}(x_i)f(x_i)+h_{xx}(x_i)g(x_i)u_i+h_{xx}(x_i)b(x_i)w_i)\delta x_i \\
&\quad +h_x(x_i)g(x_i)\delta u_i-h_x(x_i)b(x_i)w_i-(h_x(0)+h_x(0)f_x(0))\delta\overline{x_i}])\|+\epsilon_1\|\mathcal{T}(\delta u_i)\| \\
&\leqslant \|\mathcal{T}(-\delta u_i+l\,\delta u_i+(\overline{Cb})^{-1}[(h_x(x_i)+h_x(x_i)f_x(x_i)+h_x(x_i)g_x(x_i)u_i+h_{xx}(x_i)g(x_i)u_i+h_{xx}(x_i)f(x_i) \\
&\quad +h_{xx}(x_i)b(x_i)w_i)\delta x_i-h_x(x_i)b(x_i)\omega_i-(h_x(0)+h_x(0)f_x(0))\delta\overline{x_i})\|+\epsilon_1\|\mathcal{T}(\delta u_i)\| \\
&\leqslant (\epsilon+\epsilon_1)\|\mathcal{T}(\delta u_i)\|+\|(\overline{Cb})^{-1}\|(2b_{hx}+2b_{hx}b_{fx}+b_{hxx}b_f+b_{hxx}b_g\|u_i\|+b_{hxx}b_b\|w_i\|)+b_{hx}b_{gx}\|u_i\|)\|\mathcal{T}(\delta\hat{x}_i)\| \\
&\quad +\|(\overline{Cb})^{-1}\|(b_{hx}b_bb_w) \\
&= (\epsilon+\epsilon_1)\|\mathcal{T}(\delta u_i)\|+k_1\|\mathcal{T}(\delta\hat{x}_i)\|+\|(\overline{Cb})^{-1}\|(b_{hx}b_bb_w))
\end{aligned} \tag{31}
$$

where $\delta\hat{x}_i \triangleq \operatorname{argmax}_{\{\delta x_i,\delta\overline{x_i}\}}(\|\delta x_i\|,\|\delta\overline{x_i}\|)$ and $k_1=\|(\overline{Cb})^{-1}\|(2b_{hx}+2b_{hx}b_{fx}+b_{hxx}b_f+b_{hxx}b_g\|u_i\|+b_{hxx}b_b\|w_i\|+b_{hx}b_{gx}\|u_i\|)$.

We now consider the two cases $\|\delta\hat{x}_i\|=\|\delta\overline{x_i}\|$ or $\|\delta\hat{x}_i\|=\|\delta x_i\|$ separately.

Case I: $\|\delta\hat{x}_i\|=\|\delta\overline{x_i}\|$ Using Gronwall-Bellman inequality (see [16] p. 63) we write $\|\delta\overline{x_i}\|$ (the solution to (18)) in Eq. (31) in terms of $\|\delta\overline{u_i}\|$ and initial condition as shown in (32).

$$
\begin{aligned}
\|T_i(u_i)-T_i(v_i)\| \leqslant (\epsilon+\epsilon_1)\|\mathcal{T}(\delta u_i)\| + \Bigg\|\mathcal{T}\Bigg(k_1\Bigg(e^{|\overline{A}|t}\|\delta\overline{x}(0)\| \\
+\int_0^t e^{\|\overline{A}\|(t-\tau)}\|\overline{B}\|\|\delta\overline{u_i}\|d\tau\Bigg)\Bigg)\Bigg\| \\
+\|(\overline{Cb})^{-1}\|(b_{hx}b_bb_w)).
\end{aligned} \tag{32}
$$

where $\delta\overline{x}(0)$ is the initial condition of the controller (18). It can be shown that $\|\delta\overline{u_i}\|\leqslant k_2\|\delta u_i\|$. Multiplying (32) by $e^{-\lambda l}$ and defining $k\triangleq\max\{k_1k_2\|B\|,\|A\|\}$ and assuming $\lambda>k$, we have:

$$
\begin{aligned}
e^{-\lambda t}\|T_i(u_i)-T_i(v_i)\| \leqslant (\epsilon+\epsilon_1)e^{-\lambda t}\|\mathcal{T}(\delta u_i)\| \\
+\Bigg\|\mathcal{T}\Bigg(k_1\Bigg(e^{(\|\overline{A}|-\lambda)t}\|\delta x(0)\| \\
+k\int_0^t e^{-\lambda\tau}e^{(k-\lambda)(t-\tau)}\|\delta u_i\|d\tau\Bigg)\Bigg\|\Bigg) \\
+\|(\overline{Cb})^{-1}\|(b_{hx}b_bb_w)
\end{aligned} \tag{33}
$$

Case II: $\|\delta\hat{x}_i\|=\|\delta x_i\|$ Write $\|\delta x_i\|$ in terms of $\|\delta u_i\|$ using (22) and the proof follows in the same way (with $k_2=1$ in this case). Noting that the integral is strictly increasing and that for a constant $\|k\|_\lambda=k$ we obtain from (33) taking *sup* over $t\in[0,T]$:

$$
\begin{aligned}
\|T_i(u_i)-T_i(v_i)\|_\lambda \leqslant \left[\epsilon+\epsilon_1+\frac{\kappa}{\lambda-k}(1-e^{(k-\lambda)T})\right]\|\delta u_i\|_\lambda \\
+k_1\|\delta x(0)\|+\|(\overline{Cb})^{-1}\|(b_{hx}b_bb_w)
\end{aligned} \tag{34}
$$

Since $(\epsilon+\epsilon_1<1)$, we can find a $\lambda>k$ which makes $\rho_i=\epsilon+\epsilon_1+k/\lambda-\kappa(1-e^{(k-\lambda)T})<\rho<1$.

$$
\begin{aligned}
\|u_{i+1}-v_{i+1}\|_\lambda \leqslant \rho_i\|u_i-v_i\|_\lambda+b_d \leqslant \rho_i\rho_{i-1}\|u_{i-1}-v_{i-1}\|_\lambda+\rho_ib_d \\
+b_d \leqslant \rho^2\|u_{i-1}-v_{i-1}\|_\lambda+\rho b_d+b_d
\end{aligned}
$$

where b_d combines the norm bounds of the initial state errors of the controller and input disturbances. Therefore,

$$
\limsup_{i\to\infty}\|v_{i+1}-u_{i+1}\|_\lambda \to \frac{1}{1-\rho}b_d;
$$

i.e $\exists N$ such that $\forall i>N$, $u_i\in B(u^*,r)$, where u^* is the fixed point of the contraction mapping T and $B(u^*,r)$ is an open ball of radius $r=(1-\rho)^{-1}b_d$ and center u^*. If the input disturbances and initialization errors are absent, $b_d=0$ and hence u_i converges to u^*. It can be shown following Theorem 2.1 [16] that the fixed point u^* is unique.

If u_d exists, $u^*=u_d$

If $\exists u_d$ such that $P(u_d)=y_d$, the fixed point u^* of the contraction mapping $T_i(\cdot)$ is shown to be u_d in the absence of w_i and initialization error. If $u_i=u_d$, $y_i=y_d$ and $\delta y_i=y_d-y_i=0$. This implies, the output (δu_i) of the learning controller is zero. Therefore,

$$
\begin{aligned}
T_i(u_d) &= \mathcal{T}(u_d-DN|_0^{-1}N(u_d)) \\
&= u_d-\mathcal{T}(DN|_0^{-1}(y_d-y_d)) \\
&= u_d-0 \\
&= u_d.
\end{aligned} \tag{35}
$$

Hence $u^*=u_d$.

3.4 Simulation Result. In this section, we perform simulation studies with a SISO nonlinear nonminimum phase system with input disturbance described by:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix}=\underbrace{\begin{bmatrix}-x_1(t)+x_2(t)\\ -3x_2(t)+x_1^3(t)\\ x_1(t)-2x_3(t)\\ -x_4(t)+x_3^2(t)\end{bmatrix}}_{f(\mathbf{x})}+\underbrace{\begin{bmatrix}0.2+0.1\sin^2(x_4(t))\\ 0\\ 0\\ 0\end{bmatrix}}_{g(\mathbf{x})}u_i(t)+\underbrace{\begin{bmatrix}0.2\sin(40x_1(t))\\ 0\\ 0\\ 0\end{bmatrix}}_{b(\mathbf{x})}w_i(t)$$

$$y_i(t)=\underbrace{x_1(t)-3x_3(t)}_{h(\mathbf{x})} \tag{36}$$

$x(0)=0$ and the initial conditions are reset to zero after each iteration. The reference output trajectory is given by:

$$y_d(t)=\begin{cases}0.2\sin(t) & t\in[0,2\pi]\\ 0 & \text{otherwise}\end{cases} \tag{37}$$

The learning controller $DN|_0$ as given by (20) is:

$$\dot{\bar{\mathbf{x}}}_{\mathbf{i}}(t)=\underbrace{\begin{bmatrix}2 & 0 & -3 & 0\\ 0 & -3 & 0 & 0\\ 1 & 0 & -2 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}}_{\bar{A}}\bar{\mathbf{x}}_{\mathbf{i}}(t)+\underbrace{\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}}_{\bar{b}}(\delta y_i(t)+\delta\dot{y}_i(t))$$

$$\delta u_i(t)=\underbrace{[15\quad -5\quad -15\quad 0]}_{\bar{C}}\bar{\mathbf{x}}_{\mathbf{i}}(t)+\underbrace{5}_{\bar{d}}(\delta y_i(t)+\delta\dot{y}_i(t)) \tag{38}$$

$\overline{x_i}(\pm\propto)=0$. $\overline{A}$ is unstable. Hence we apply noncausal stable inversion to solve the above differential equation. First we consider the input random noise is absent $(w_i=0)$. The simulation result shows near perfect tracking of the desired trajectory Fig. 3. A learning curve is plotted in Fig. 4 to show the convergence of the norm of the output error $\|y_d-y_{\text{out}}\|$.

Next we introduce w_i as a nontrivial bounded input disturbances. Simulation (Fig. 5) shows approximate tracking of the desired trajectory after a couple of iterations. w_i is normally distributed random numbers bounded between ±1. We have passed the white noise through a low pass filter to ensure continuity.

4 Nonlinear Nonminimum Phase Plant With Repetitive Input Disturbances

In this section, we apply the same learning algorithm to a plant with repetitive input disturbances. To model input repetitive disturbances, the plant equation (11) is modified as:

$$\dot{x}_i(t)=f(x_i(t))+g(x_i(t))(u_i(t)+\Delta u(t)),\quad x_i(\pm\infty)=0 \tag{39}$$

$$y_i(t)=h(x_i(t));$$

where $\Delta u(t)\in\mathbb{R}$. The function $\Delta u(t)$ represents unmodelled dynamics or input disturbance which repeats at every iteration of the ILC system. Since Δu remains the same at every iteration a learning controller can use the information about the disturbance from the previous execution to improve the tracking performance in the next execution. We impose the first five assumtions as discussed in Section 3.1 on the given nonlinear system. Assumption 6 is replaced by Assumption $6'$ which states that:

$(\mathcal{A}6')$ The disturbance $\Delta u(\cdot)$ is continuous and bounded by b_u (i.e., $\|\Delta u(t)\|\leq b_u$).

It is proved in Appendix B that perfect tracking of the desired trajectory is achieved in this case. Also Matlab simulation (Fig. 6) shows perfect tracking in presence of the same disturbance at every iteration of learning.

4.1 Simulation Result. In this section, we perform simulation studies with a SISO nonlinear nonminimum phase system with repetitive disturbance:

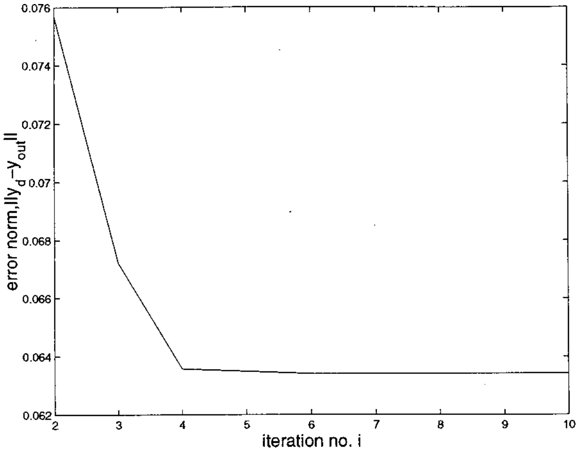

Fig. 4 Learning curve to show convergence

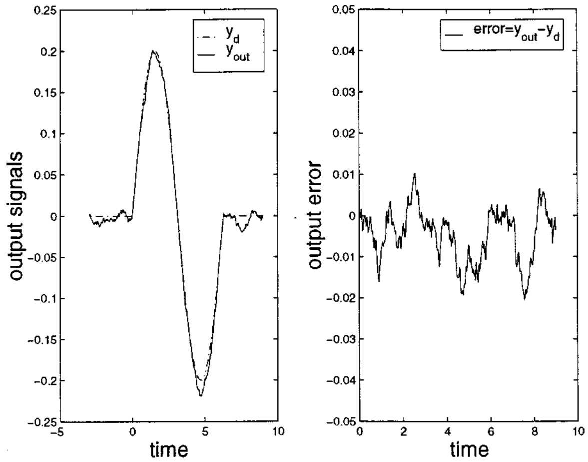

Fig. 5 Tracking of the same system in presence of input disturbance after 3 iterations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -x_1(t)+x_2(t) \\ -3x_2(t)+x_1^3(t) \\ x_1(t)-2x_3(t) \\ -x_4(t)+x_3^2(t) \end{bmatrix}}_{f(\mathbf{x})} + \underbrace{\begin{bmatrix} 0.2+0.1\sin^2(x_4(t)) \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g(\mathbf{x})} (u_i(t)+\Delta u(t))$$

$$y_i(t) = \underbrace{x_1(t)-3x_3(t)}_{h(\mathbf{x})} \tag{40}$$

$x(0)=0$ and the initial conditions are reset to zero after each iteration. The reference output trajectory is given by:

$$y_d(t) = \begin{cases} 0.2\sin(t) & t\in[0,2\pi] \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

The learning controller $DN|_0$ is given by (38).

The learning controller discussed in this paper can be applied to a generic class of nonminimum phase plants. However, the controller design requires the precise knowledge of the linearized plant. In [18] we present results that allow a inversion based ILC for plants with uncertainties. A proof of convergence is presented along with the condition for convergence that quantify the perturbation allowable for the learning algorithm to converge. The performance of the approach is also illustrated with simulation results.

5 Conclusion

A stable inversion based learning controller for time-invariant nonlinear nonminimum phase systems is presented. It guarantees learning under quite general assumptions. Theoritical assertions arc corroborated by simulation results which demonstrate that in the presence of random input disturbances the tracking error is uniformly bounded. It is proved that the bound on the tracking error is a continuous function of the bound on the input noise. It has been also proved that if u_i is replaced by $u_i+\Delta u$, where Δu is an unknown function which lumps together various unstructured uncertainties due to parameter uncertainty, model simplification, unmodelled dynamics, repetitive disturbances and so on (see [16], p 578), perfect tracking of the desired trajectory is achieved in the absence of random input disturbance (i.e., $w_i=0$). Several modifications can be made on the developed algorithm to improve its performance. For example, if at any iteration step, the update law produces an input that is outside the allowable convex set within which the input is constrained to lie, then the projection of the produced input on this convex set must be used instead, in the next iteration of learning, as pointed out in [9]. We further conjecture that the proposed learning algorithm can be readily extended to slowly time-varying nonlinear systems. Though the class of systems considered here is fairly general, the effect of output sensor noise has not been considered. In practice, derivatives cannot be reliably computed in the presence of output sensor noise. Furthermore, the plant may itself produce an output signal that is not differentiable. This will constitute an interesting area for future research.

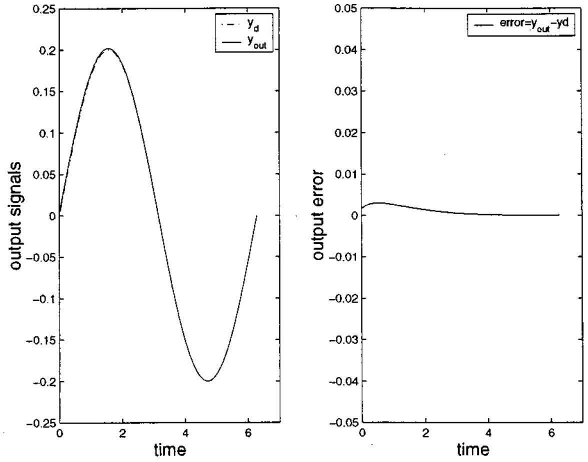

Fig. 6 Tracking of nonlinear nonminimum phase system with repetitive disturbance after 5 iterations

Nomenclature

$\|\cdot\|$ = Euclidean norm

$\|f\|_{\infty,[0,T]}$ = $\sup_{t\in[0,T]}\|f(t)\|$

$L_\infty[0,T]$ = the space of all functions such that $\|f\|_{\infty,[0,T]}<\infty$ on the closed interval $[0,T]$

L_∞ = the space of all functions such that $\|f\|_\infty<\infty$ on $(-\infty,\infty)$

$C^0[0,T]$ = the space of all continuous functions on the closed interval $[0,T]$

$C^r[0,T]$ where $(r\geq 1)$ = the space of all r times continuously differentiable functions on the closed interval $[0,T]$

$B_r[0,T]$ = $\{\overline{u}(\cdot)|\overline{u}(t)\in\mathbb{R}^r$ and $\|\overline{u}(\cdot)\|<r<\infty\}$ on the closed interval $[0,T]$

$o(h)$ = a function $g(h)$ is $o(h)$, implies $\lim_{\|h\|\to 0}\|g(h)\|/\|h\|=0$.

Appendix A

Sketch of the Proof of Lemma 1. Consider a nonlinear system given by:

$$\begin{aligned} \dot{x} &= f(x)+g(x)u \\ y &= h(x) \end{aligned} \tag{42}$$

with a well-defined relative degree $r\leq n$. Let the *normal form* of the above system be given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \vdots &= \vdots \\ \dot{\xi}_r &= b(\xi,\eta)+a(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi_1 \end{aligned} \tag{43}$$

First we need to show that the linear approximation of equations in normal form coincides with the normal form of the linear approximation of the original description of the system and this amounts only to show that the relative degree of the system and that of its linear approximation are the same. To this end, suppose that the vector field $f(x)$ has an equlibrium at $x=0$, i.e., $f(0)=0$, and consider for $f(x)$ an expansion of the form

$$f(x)=Ax+f_2(x) \quad \text{with} \quad A=\left[\frac{\partial f}{\partial x}\right]_{x=0} \qquad \text{and} \qquad \left[\frac{\partial f_2}{\partial x}\right]_{x=0}=0$$

which separates the linear approximation Ax from the higher-order term $f_2(x)$. Similarly expand $h(x)=Cx+h_2(x)$ where $h(0)=0$: $C=[\partial h/\partial x]_{x=0}$ and $[\partial h_2/\partial x]_{x=0}=0$. Also expand $g(x)$ as $g(x)=B+g_1(x)$ with $B=g(0)$. Therefore, the linear approximation of the system at $x=0$, is defined as $\dot{x}=Ax+Bu$: $y=Cx$. It can be shown, by induction, that

$$L_f^k h(x)=CA^k x+d_k(x): \text{ where } d_k(x)\text{is such that}$$

$$[\partial d_k/\partial x]_{x=0}=0.$$

From this it can be deduced that

$$CA^kB=L_gL_f^kh(0)=0 \quad \text{for all } k<r-1$$

$$CA^{r-1}B=L_gL_f^{r-1}h(0)\neq 0$$

i.e. the relative degree of the linear approximation of the system at $x=0$ is exactly r. From this fact it can be concluded that taking the linear approximation of the equations in the normal form (43) based on expansions of the form:

$$b(\xi,\eta)=R\xi+S\eta+b_2(\xi,\eta)$$

$$a(\xi,\eta)=K+a_1(\xi,\eta)$$

$$q(\xi,\eta)=P\xi+Q\eta+q_2(\xi,\eta)$$

yields a linear system in normal form

$$\begin{aligned}\dot{\xi}_1&=\xi_2\\ \dot{\xi}_2&=\xi_3\\ \vdots&=\vdots\\ \dot{\xi}_r&=R\xi+S\eta+Ku\\ \dot{\eta}&=P\xi+Q\eta\\ y&=\xi_1\end{aligned} \tag{44}$$

It is shown by [19] that the eigenvalues of Q coincide with the zeros of the linear system given by (44). Also note that the Jacobian matrix $[\partial q/\partial\eta]_{(\xi,\eta)=0}$ describes the linear approximation at $\eta=0$ of the zero dynamics of the original nonlinear system (43). Therefore, if a nonlinear system is nonminimum phase (linearized zero dynamics has atleast one unstable eigenvalue): the zeros of the transfer function of the linear approximation of the original system at $x=0$ are not all stable.

Appendix B

Convergence Analysis in Presence of Repetitive Disturbance. From the stable nonlinear system (39) we define an input-to-output nonlinear map P as follows:

$$P:\tilde{u}_i\mapsto y_i,$$

where $\tilde{u}_i\triangleq u_i+\Delta u$ (a computed input). Define a nonlinear operator $\tilde{N}$ as:

$$\begin{aligned}N\triangleq-(DToP):\tilde{u}_i&\mapsto-(\delta y_i+\delta\dot{y}_i)\\ L_\infty&\to L_\infty.\end{aligned} \tag{45}$$

Denoting $\delta x_i=x_d-x_i$, $\delta y_i=y_d-y_i$ and $\delta u_i=u_d-u_i-\Delta u$, we derive (by Taylor series expansion) a linearized plant from (13) as follows:

$$\begin{aligned}\dot{x}_i(t)+\delta\dot{x}_i(t)&=f(x_i(t)+\delta x_i(t))+g(x_i(t)+\delta x_i(t))\cdot[u_i(t)\\ &\quad+\delta u_i(t)+\Delta u];\\ &\approx f(x_i(t))+f_x(x_i(t))\delta x_i(t)+[g(x_i(t))\\ &\quad+g_x(x_i(t))\delta x_i(t)][\tilde{u}_i(t)+\delta u_i(t)];\end{aligned} \tag{46}$$

(By Taylor series expansion and neglecting h.o.t) where

$$f_x(x_i(t))\triangleq\frac{\partial f}{\partial x}(x_i(t)); \quad g_x(x_i(t))\triangleq\frac{\partial g}{\partial x}(x_i(t));$$

$$\tilde{u}_i(t)=u_i(t)+\Delta u$$

Also, $y_i(t)+\delta y_i(t)=h(x_i(t)+\delta x_i(t))$.

Subtracting (39) from (46) and neglecting h.o.t we obtain the linearized plant (around the solution $x_i(t)$ to (39)) as:

$$\begin{aligned}\delta\dot{x}_i(t)&=f_x(x_i(t))\delta x_i(t)+g(x_i(t))\delta u_i(t)\\ &\quad+g_x(x_i(t))\delta x_i(t)\tilde{u}_i(t), \quad \delta x_i(0)=0\\ \delta y_i(t)&=h_x(x_i(t))\delta x_i(t);\end{aligned} \tag{47}$$

Since (18) is stable, it can be proved by Lyapunov methods that (47) is also BIBO stable if x_i lies within a certain bound. Note that, here also we can replace $\delta\tilde{x}_i(0)=0$ (as in 18) with $\delta\tilde{x}_i(\pm\infty)=0$ and not alter the I-O map. Now replacing $u_i(t)$ by $\tilde{u}_i(t)$ it can be proved in the same way as in Section 3.3 that the sequence of input converges to $u^*(t)$.

Appendix C

Proposition 1. Consider the nonlinear plant P described by the following differential equation

$$\dot{x}=f(x,u); \quad x(0)=x_0 \tag{48}$$

on the interval [0,T] where $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ *is smooth in x and u, and* $\|u\|_\infty=\sup_{t\in[0,T]}\|u(t)\|_2<M$; *T is chosen such that solutions exists on* $[0,T]\forall u\in L_\infty$ *satisfying the bound. Let* $\phi_u(t,x_0)$ *denote the solution of the differential equation on [0,T]. The Frechet derivative* $D_u\phi_u(t,x_0):L_\infty[0,T]\to L_\infty[0,T]$ *of the map* $u(\cdot)\mapsto\phi_u(\cdot,x_0)$ *is given by*

$$D_u\phi_u(t,x_0)\zeta=DP|_u\zeta=\int_0^t\Phi(t,\tau)\frac{\partial f}{\partial u}[\phi_u(\tau,x_0),u(\tau)]\zeta(\tau)d\tau \tag{49}$$

where Φ(t,τ) is the state transition matrix for

$$\dot{x}=\left[\frac{\partial f}{\partial x}(\phi_u(t,x_0),u(t))\right]x. \tag{50}$$

Proof: See [17] for proof.

Appendix D

Boundary Value Problem for Nonminimum Phase Systems. A nonlinear nonminimum phase system can be viewed as a mapping of $C_{[0,\infty]}$ to $C_{[0,\infty]}$ or as a mapping from $L_{[-\infty,\infty]}$ to $L_{[-\infty,\infty]}$. In the first case the inverse mapping is unbounded, while in the second, it is bounded but noncausal. It is the second view that enables a proper perspective on tracking control problems as feedforward need not be computed causally from sensor outputs.

If a nonlinear plant with hyperbolic zero dynamics is nonminimum phase, the inverse of the linearized plant is unstable. Hence we perform stable noncausal inversion of the linearized plant to obtain the learning controller for ILC scheme described in Section 3. An essential element to solving this problem is finding a solution meeting boundary conditions at $\pm\infty$. Hence for the linear learning controller this reduces to finding solutions to

$$\dot{x}=Ax+Bu; \quad x(\pm\infty)=0. \tag{51}$$

where we assume that A has no $j\omega$-axis eigenvalues and $u \in L_1 \cap L_\infty$. Without loss of generality, assume that A is block diagonal:

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}$$

where A_- and $-A_+$ are both Hurwitz. It is easily verifiable by substitution into (51) (using the notation $1(t)$ for the unit step function), a bounded state-transition matrix can be defined by

$$\phi(t) = \begin{bmatrix} 1(t)e^{A_- t} & 0 \\ 0 & -1(-t)e^{A_+ t} \end{bmatrix}, \tag{52}$$

and that the solution to Eq. (51) meeting the boundary conditions $x(\pm\infty) = 0$ has the form

$$x(t) = \int_{-\infty}^{+\infty} \phi(t-\tau) B u(\tau) d\tau \tag{53}$$

Define a mapping $\mathcal{A}: L_1 \cap L_\infty \to L_1 \cap L_\infty$ as the mapping taking u to x in Eq. (51). That is $\mathcal{A}: u \mapsto x$

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