Observers for Linear Systems with Quantized Outputs

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Abstract

State estimation via nonlinear observers is described for a linear system with quantized outputs. Although there may be significant quantization error on average, it is possible to design observers with exponentially stable tracking error. Here we show that a Luenberger Observer modified, so that feedback is impulsive at the quantizer transitions, is one such observer.

keywords: hybrid control

1 Introduction

Linear systems with quantized outputs are as common as digital controllers since all digital control systems employ digital-to-analog converters with finite precision. The quantization effects are only relevant, however, when quantization errors are large relative to other error and noise sources. For example, quantization is an issue in precision motion control systems employing motor-encoder pairs, in systems with limit-switch outputs, or any system where continuous states are measured by digital means. On the other hand, careful modeling of quantization is not an issue in communications systems that operate at high noise levels.

The work described herein was also motivated by issues in hybrid-control—systems that have both continuous-time and discrete-time character. For the purpose of understanding hybrid control, linear systems with quantized outputs serve as a simple, yet challenging, prototype.

Much of the work on quantization in control systems seeks to simply bound the performance degradation caused by the use of quantizers. Early work by [11] takes this approach. Curry, in his thesis [4] develops maximum-likelihood estimates for static linear systems driven by Gaussian noise and having quantized outputs. When Curry addresses dynamic linear systems, his exact methods become intractable and he resorts to more traditional approximations where quantization is modeled as noise. More recent and fundamental work on systems with quantized outputs is described in [2].

Another approach is taken by Schweppe [9] in which an ellipsoidal set estimating state and the state uncertainty is propagated forward in time. The performance calculation is intractable analytically in this case also. More recently, Miller, Michel, and Farrel [10] establish useful bounds on tracking in digitally controlled plants were there is numerical quantization in the control computation. Along similar lines, a book by Gevers and Li looks at numerical computation in systems in control with quantization.

In contrast to the approach above, we attempt in this paper an exact analysis of quantization. Quantization is a deterministic nonlinearity, and, by treating it as such, we obtain excellent state tracking performance in an observer. This exact treatment of the quantization nonlinearity is similar in spirit to the work of Delchamps [5] [6] [7] [8]. Delchamps work is fundamental to the feedback problem in discrete time systems, whereas we focus on the continuous time problem and observers. More recent work addressing chaos in feedback systems with quantization is due to Stepan and Haller [3].

We apply our result to the example problem of state estimation for a motor-encoder pair. This is the first continuous-time observer for the motor-encoder problem exhibiting exact state tracking, but it is important to realize that industrial designers have resolved many of the challenges with this problem by measuring the interarrival time between encoder transitions to estimate motor angular velocity.
2 Problem Statement

We consider the observer design problem for a SISO linear time-invariant system with quantized outputs:

\begin{align}
\dot{x} &= Ax + Bu \\
y &= Cx \\
y_q &= Q[y].
\end{align}

We have taken the feedthrough term \( D = 0 \) for simplicity although it is straightforward to incorporate. The quantizer nonlinearity \( Q[\cdot] \) in equation 3 is taken to be the round-off nonlinearity of figure 1. A typical approach to designing an observer for the system above is to simply ignore the quantizer. For example a Luenberger observer has the form

\begin{align}
\dot{x} &= A\hat{x} + bu - L(\hat{y} - y), \quad \hat{x}(t_0) = \hat{x}_0 \\
y &= C\hat{x}
\end{align}

The state estimation error dynamics of the linear observer with quantizer is given by

\begin{align}
\dot{e} &= Ae - L(\hat{y} - y_q), \quad e(t_0) = e_0 \\
&= Ae - L(\hat{y} - y - \Delta y_q) \\
&= (A - LC)e + L\Delta y_q
\end{align}

where \( e = x - \hat{x}, \Delta y_q = y_q - y \), and \( L \) is designed so that \( A - LC \) is Hurwitz. Thus, there exists an \( M = M^T > 0, \quad Q = Q^T > 0 \) such that \( (A - LC)^T M + M(A - LC) = -Q \). \( V(t, e) = e^T Me \) is a Lyapunov function for the standard Luenberger Observer. The time derivative of the Lyapunov function is

\[ \dot{V}(t, e) = 2e^T M \Delta y_q - e^T Q e \]  (7)

Since the quantizer error can be as much as \( 1/2 \), the estimation error can only be shown to converge to a neighborhood of zero since the first term in equation 7 has arbitrary sign and can only be bounded. The problem, then, is to redesign the observer, and to find conditions under which the error tends to zero.

3 The Impulsive Luenberger Observer

Our observer designs are guided by a simple idea: at quantizer transition the value of \( Cx \) is known exactly. The simplest observer is a variation of the Luenberger observer in which \( y \) is approximated by an sequence of impulses at the quantizer transition times. Specifically,

\[ y \approx \sum_{k=1}^{\infty} y(t_k) \Delta t_k \delta(t - t_k) \]  (8)

where \( t_k \) are the quantizer transition times, and \( \Delta t_k := t_k - t_{k-1} \) are the transition interarrival times. Observe that the right-hand side of equation 8 depends on the values of \( y \) at the quantizer transitions which are known. For simplicity define \( y_k = y(t_k) \). Looking at figure 1, \( y_k = y_{n,n+1} \) when the quantizer output switches from \( n \) to \( n+1 \) at time \( t_k \) so that \( y(t_k) \) is known exactly. Replacing \( y \) with \( \sum_{k=1}^{\infty} y(t_k) \Delta t_k \delta(t - t_k) \) in equation 5 leads to an observer with discrete updates at quantizer transitions. We call this hybrid system the Impulsive Luenberger Observer:

\begin{align}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t), \quad t \notin \{t_k\}_{k=0}^{\infty} \\
\hat{x}(t) &\leftarrow \hat{x}(t) - \Delta t_k L(\hat{y} - y_k), \quad t \in \{t_k\}_{k=0}^{\infty}
\end{align}

of the right-hand side. Let \( e(t) = x(t) - \hat{x}(t) \) be the estimated error. Then we have

\[ \dot{e}(t) = Ae(t), \quad e(t_0) = e_0, \quad \forall \ t \notin \{t_k\}_{k=0}^{\infty} \]

\[ e_{new}(t) \leftarrow (I - \Delta t_k LC)e(t), \quad \forall \ t \in \{t_k\}_{k=0}^{\infty} \]  (10)

which defines the evolution of the observer error. We now proceed to show convergence of the Impulsive Luenberger Observer for the case that \( \Delta t_k \) are uniformly bounded by a sufficiently small number.
4 Stability of the Impulsive Luennberger Observer

If the interarrival times of the quantizer transitions were constant then the stability analysis of the observer error dynamics would be a simple discrete time problem. When the interarrival times are nonuniform, there is some labor involved. First, recall the definition of $O(t^n)$.

**Definition 1** Let $f(t) \in R$ (or $R^n$, or $R^{n \times n}$). We say that $f(t) = O(t^n)$ if there exists $\epsilon > 0$, $K > 0$ such that $|t| < \epsilon \rightarrow \|f(t)\| \leq K|t^n|$ for all $t \in [-\epsilon, \epsilon]$.

The stability of the observer error in the Impulsive Luennberger Observer is proved in the following.

**Theorem 1** Suppose that $\Delta t_k \in (0, \epsilon)$ for all $k$, for some $\epsilon$ sufficiently small, and that $A - LC$ is Hurwitz. Then the observer error dynamics of the Impulsive Luennberger Observer (equation 10) are exponentially stable.

**Proof:** From 10 and using the notation $e^{(A-LC)\Delta t_i}$ to indicate that a term is removed, we have

$$e(t) = e^{A(t-t_k)} \prod_{i=1}^{k} (I - \Delta t_i LC) e^{A\Delta t_i} e_0$$

$$= e^{A(t-t_k)} \prod_{i=1}^{k} (e^{(A-LC)\Delta t_i} + O(\Delta t_i^2)) e_0$$

$$= (I + O(\epsilon)) e^{(A-LC)(t-t_k)} \times e^{(A-LC)\Delta t_k} + \sum_{j \neq k} e^{(A-LC)\Delta t_j} \cdots e^{(A-LC)\Delta t_k} O(\Delta t_j^2)$$

$$\cdots e^{(A-LC)\Delta t_1} + \sum_{j < k} e^{(A-LC)\Delta t_k} \cdots e^{(A-LC)\Delta t_j} O(\Delta t_j^2)$$

$$\cdots e^{(A-LC)\Delta t_1} O(\Delta t_j^2) \cdots e^{(A-LC)\Delta t_1} O(\Delta t_j^2)$$

$$\cdots e^{(A-LC)\Delta t_1} + \sum_{j < k < m} e^{(A-LC)\Delta t_1} \cdots e^{(A-LC)\Delta t_m} O(\Delta t_m^2)$$

where the indices $j, l, m, p$ in the summations run from 1 to $n$. If we use the the facts that $\sum_{i=1}^{k} \Delta t_i = t_k$, $t = t_k + t - t_k$, and $O(\Delta t_k^2)$ commutes with matrices we have

$$e(t) = (I + O(\epsilon))[e^{(A-LC)t} + \sum_j e^{(A-LC)(t-\Delta t_j)} O(\Delta t_j^2)$$

$$+ \sum_{j < k} e^{(A-LC)(t-\Delta t_j-\Delta t_k)} O(\Delta t_j^2) O(\Delta t_k^2)$$

$$+ \sum_{j < k < m} e^{(A-LC)(t-\Delta t_j-\Delta t_k-\Delta t_m)} O(\Delta t_j^2) O(\Delta t_k^2) O(\Delta t_m^2)$$

$$\times \sum_{j < k < m} e^{(A-LC)(t-\Delta t_j-\Delta t_k-\Delta t_m)} O(\Delta t_j^2) O(\Delta t_k^2) O(\Delta t_m^2) + \cdots]e_0$$

(12)

Since $e^{-(A-LC)\Delta t_j} = I + O(\Delta t_j)$

and $O(\epsilon) + O(\epsilon^2) = O(\epsilon)$

equation 12 can be written as

$$e(t) = (I + O(\epsilon)) e^{(A-LC)t} [I + \sum_j O(\Delta t_j^2)]$$

$$+ \sum_{j < k} O(\Delta t_j^2) O(\Delta t_k^2) + \cdots]e_0$$

(16)

Taking note of

$$\sum_j O(\Delta t_j^2) = \sum_j \Delta t_j O(\Delta t_j)$$

$$\leq \sum_j \Delta t_j O(\epsilon) = O(\epsilon) \sum_j \Delta t_j$$

$$\leq O(\epsilon)t$$

(17)

also

$$\sum_j O(\Delta t_j^2) O(\Delta t_k^2) \leq \sum_j O(\Delta t_j^2) \sum_i O(\Delta t_i^2)$$

$$\leq \frac{O(\epsilon)^2t^2}{2!}$$

(18)
and
\[
\sum_{j<i<m} O(\Delta t_j^2)O(\Delta t_i^2)O(\Delta t_m^2) \\
\leq \sum_{j} O(\Delta t_j^2) \sum_{i} O(\Delta t_i^2) \sum_{m} O(\Delta t_m^2) \\
\leq \frac{(O(\epsilon))^3 t^3}{3!} \quad (19)
\]

The error dynamics becomes
\[
e(t) = (I + O(\epsilon))e^{(A - LC)t} \\
\times (I + O(\epsilon)t + \frac{(O(\epsilon))^2 t^2}{2!} + \frac{(O(\epsilon))^3 t^3}{3!} \ldots )e_0 \\
= (I + O(\epsilon))e^{(A - LC)t}e_0 \\
= (I + O(\epsilon))e^{(A - LC + O(\epsilon))t}e_0 \quad (20)
\]

Since the eigenvalues of a matrix are continuous functions of the matrix and \( A - LC \) is Hurwitz, for \( \epsilon \) sufficiently small we have \( e(t) \to 0 \) exponentially as \( t \to \infty \).

5 Velocity Estimation of a DC-Motor with Optical Encoder

As an example application, consider a simple sinusoidally driven DC-motor with optical encoder. We apply the Impulsive Luenberger Observer and use the output of a shaft encoder as a quantized measurement of the shaft angle. In commercial robots velocity estimates for feedback are extracted from the encoder output. If the quantization is ignored, the angular velocity can be estimated by using a standard Luenberger Observer. We consider the simple DC-motor system with an optical encoder as shown in figure 2. This system can be described by

\[
\dot{x} = Ax + Bu \\
y = Cx \\
y_q = Q[y]
\]

where \( x = [\theta, \omega] \), \( A = \begin{bmatrix} 0 & 1 \\ 0 & -K_1 K_2 / JR \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ K_1 / JR \end{bmatrix} \), \( C = [1, 0] \), and \( \theta \) is angular position, \( \omega \)

is angular velocity, and \( K_1, K_2, J, R \) are known coefficients of the DC-motor. The quantized output from the optical encoder on the motor shaft can be modeled as \( y_q = Q[(y)] \), and is used to calculate estimator states \( \hat{x}_{new}(t_k) \) in order to reset for motor observer using the results of the last section. For purposes of simulation we use the coefficients \( K_1 K_2 / JR = 1.5121 \), \( K_1 / JR = 1.012 \) of DC-motor. The simulation results shown in figure 3 support the theoretical results of stability.

6 Conclusion

Linear systems with quantized outputs have proved to be very interesting hybrid systems in that exponentially stable observers, also of hybrid structure, have been shown to exist. In this paper the simple "Impulsive Luenberger Observer" has been analyzed and simulated with excellent results. Related observers have
been studied in [1]. This problem remains largely open to further study, especially with regard to stability analysis methods for certain observers employing sliding modes (also in [1]).

References


