Control of 2-periodic Motion for Bouncing Ball

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Abstract
Controlling the height of a bouncing ball on a vibrating plate is a prototype problem for the control of nonlinear periodic motions. Further, analysis of the bouncing ball system is convenient in that a discrete-time system can be derived from the continuous dynamics without the explicit construction of a Poincaré map. If the plate vibrates sinusoidally with a particular frequency, only certain bounce-heights can be achieved in periodic motion. In this paper we describe a control scheme wherein the instantaneous frequency of the vibrating plate is modulated, so that arbitrary controlled bounce-heights are achieved in stable periodic motion involving two bounces per period. The base carrier frequency of the plate is held constant.

1 Introduction
Repeated impacts of a ball bouncing on a sinusoidally vibrating plate has been studied in [1]. A standard assumption is that the plate vibration amplitude is small so that the "bounce-to-bounce" dynamics are described by a simple two dimensional discrete-time map. The discrete-time mapping contains a horse-shoe in general, so that the ball motion is typically chaotic. As the frequency, \( \omega \), of the plate is increased gradually from zero, successive bifurcations in the bounce-to-bounce mapping are obtained; let's call the bifurcation frequencies \( \omega_0, \omega_1, \omega_2, \ldots \). The bifurcation at \( \omega_0 \) is of saddle-node type [1], and for every \( \omega \in (\omega_0, \omega_1) \), there exist a pair of fixed points, corresponding to 1-periodic motions of the ball. At the second bifurcation \( \omega_1 \), a period doubling or flip bifurcation occurs, whereas after the stable 1-periodic fixed point becomes unstable, and a pair of stable 2-periodic fixed points appear instead. At \( \omega_2 \), a second flip bifurcation occurs, yielding stable orbits of period 4 and unstable orbits of period 1 and 2. This process continues so that between any two consecutive bifurcations frequencies \( \omega_k \) and \( \omega_{k+1} \) stable orbits of period \( 2^k \) and unstable orbits of periods \( 2^k \), \( 0 < l < k \) are obtained. The intervals \( \omega_k - \omega_{k-1} \) decrease with increasing \( k \), in such a way that an accumulation point is ultimately reached. In the framework of the dynamics outlined above, in this work we study the control of stable 2-periodic motions.

Vincent [2] shows that any of the stable 1-periodic orbits at a desired plate frequency can be reached from an arbitrary initial state. The frequency of the plate vibration is used as the control input. An open-loop control initially drives the plate at a relatively high frequency so that the ball motion is chaotic. And then, when the ball motion enters the domain of attraction of the desired stable fixed point, the frequency is switched back to the desired value. If that fixed point is not a stable one, a closed loop controller must be switched on at the same time, to stabilize the ball motion to the desired (phase,height) combination of the equilibrium state. Note that, the bounce-height attainable in this approach remains fixed for any particular frequency.

It is possible to bounce the ball to \( m \) different heights in \( m \times n \) cycles of the plate before repeating the pattern. In such a process, fixed points of the map at a given frequency correspond to a particular set of \( m \) bounce-heights of the ball. If we apply frequency modulation (FM) to the plate motion, we observe that it is possible to vary the \( m \) periodic bounce-heights of the ball by varying the modulation index \( \beta \). The bounce heights are continuous functions of the parameter \( \beta \). It is explained in section 4.1 why the frequency modulation method is not applicable for changing the bounce heights for 1-periodic motions of the bouncing ball. For the range of \( \beta \) chosen as the control input, in our simulations, it has been observed from the linearized system that the fixed points corresponding to the 2-periodic motions are stable. Otherwise, it would be necessary to design a feedback controller so that the equilibrium solution has the desired stability properties.

2 A Simplified Model for the Bouncing Ball
Let the motion of the plate be given by \( y(t) = A_x \cos(\omega_c t) \), where \( y(t) \) is the displacement of the plate in the vertical direction, \( A_x \) is the amplitude of the plate, \( \omega_c \) is the frequency of the plate, and \( t \) is the time. If \( U, V, \) and \( W \) are the absolute velocities of the approaching ball, the departing ball, and the table \( W(t) = -A_x \omega_c \sin(\omega_c t) \) respectively, the usual impact relation can be written as [2]:

\[
V(t_0) = \left( \frac{m}{\omega_c} \right) U(t_0) + \left( \frac{1}{\omega_c} \right) W(t_0),
\]

where \( m \) is the mass ratio of the ball to the plate, and \( 0 < c \leq 1 \) is the coefficient of the restitution. \( t_0 \) is the time of the \( j \)th impact. If we assume that the distance the ball travels between impacts under the influence of gravity, \( g \), is large compared to the overall displacement of the table, then the time interval between the impacts can be easily approximated as: \( t_{j+1} - t_j = \frac{2V(t_j)}{g} \), with the velocity of approach at \( (j+1) \)th impact being:

\[
U(t_{j+1}) = -V(t_j).
\]

Combining the above equations we obtain the recurrence relationship relating the velocity of the ball after the \( (j+1) \)th impact to the \( j \)th impact as follows:

\[
V(t_{j+1}) = -a_2 V(t_j) - a_1 A_x \omega_c \sin(\omega_c t_j) + \frac{2V(t_j)}{g},
\]

where \( a_1 = \left( \frac{1}{\omega_c} \right) \) and \( a_2 = \left( \frac{m}{\omega_c} \right) \). Nondimensionalizing (1), and substituting \( \tilde{\alpha}_1 = \frac{a_1 A_x}{g} \), \( \tilde{\phi}_1 = \omega_c t_j \) and \( \tilde{\psi}_1 = \frac{2V(t_j)}{g} \), the state space equations can be written in the form of a nonlinear map \( f \) as follows:

\[
f : (\phi_j, \psi_j) \mapsto (\phi_{j+1}, \psi_{j+1}) = \frac{2V(t_j)}{g},
\]

\[
\phi_{j+1} = \phi_j + \psi_j, \quad \phi_{j+1} = -a_1 A_x \omega_c \sin(\phi_j + \psi_j);
\]

3 2-Periodic Solutions
It has been shown in [1] that the bouncing ball can exhibit periodic motions of all periods. For example, the ball can
bounce to a fixed height at every $n$ cycles of the plate. It can also bounce to $m$ different heights at $n \times m$ cycles of the plate before repeating the pattern. In such cases $\psi_j + m = \psi_i$ and $\psi_j + m = \psi_i$ must be satisfied for all $j$, and we refer to such periodic motion of the ball as $m$-periodic. In this work, we have considered only the $n = 1$ case, and unless otherwise noted, "m-periodic" in the rest of the paper refers to the $n = 1$ case. The primary focus in this work is on the 2-periodic motion (Fig 8a), though the general principle can be readily applied to higher periods.

In order to find the fixed points of (2) corresponding to 1-periodic motion, the conditions that are required to satisfy are $\psi_{j+1} = \psi_j$ as well as $\psi_{j+2} = \psi_j$. Also the right hand side of the first equation of (2) must be evaluated modulo $2\pi$. Solving the above equations, the fixed points for the 1-periodic motion are obtained as:

$$\psi_n = 2n\pi \text{ and } \phi_n = \arcsin \left( \frac{2\pi n + (1 + a_2)}{-a_1\omega^2} \right)$$

where $n = 0, \pm 1, \pm 2, \ldots, \pm N$, is the number of cycles of the plate between two consecutive bounces and $N$ is the greatest integer such that $2N\pi(1 + a_2) < a_1\omega^2$. The maximum bounce height is given by $h = \frac{2n\pi}{\omega a_1\omega^2}$. The Jacobian matrix of the linearized system when evaluated at the fixed point $(\phi_n, \psi_n)$ is

$$J = \begin{bmatrix} 1 & -a_2 & \cos(\phi_n + \psi_n) \\ \gamma & -a_2 & \cos(\phi_n + \psi_n) \end{bmatrix}$$

where $\gamma = \frac{a_2}{a_1} \omega^2$. The eigenvalues of (4) are given by $\lambda_{1,2} = \frac{1}{2} \left[ (1 - a_2 + \gamma^2) \pm \sqrt{(1 - a_2 + \gamma^2)^2 + 4a_2} \right]$. Substituting $(\phi_n, \psi_n)$ from (3), we find that there exists an eigenvalue of $1$ for $\gamma = -2\pi n(1 + a_2)$ i.e. at $\omega_n = \sqrt{2\pi n(1 + a_2)}$. It is verified that at $\omega_n = \omega_0$ the map $f$ undergoes a saddle-node bifurcation.

From (4) and the eigenvalues we find that corresponding to $\gamma = 2\pi/\omega^2(1 + a_2)^2 + (1 - a_2)^2$, there exists an eigenvalue of $-1$, and hence the fixed point is nonhyperbolic. Moreover, at the associated frequency $\omega_1$, all the other conditions as derived in [3] for period-doubling bifurcation or flip bifurcation are satisfied by the map $f$. Hence $f$ undergoes a period-doubling bifurcation at the frequency $\omega_n$. From (2), we can write the state-space equations for the 2-step motion (i.e. the states at $(j + 2)$th impact in terms of the states at the $j$th impact) as follows:

$$\phi_{j+2} = \phi_j + \psi_j - a_2 \psi_j - a_1 \omega^2 \sin(\phi_j + \psi_j)$$

$$\psi_{j+2} = a_2^2 \psi_j + a_2 a_1 \omega^2 \sin(\phi_j + \psi_j) - a_1 \omega^2 \sin(\phi_j + \psi_j)$$

In order to find the equilibrium solutions for 2-periodic motion from (5), the following conditions must be satisfied:

$$\phi_{j+2} = \phi_j \text{ and } \psi_{j+2} = \psi_j$$

where the right hand side of the first equation must be evaluated modulo $2\pi$. Linearizing (5) about the 2-periodic fixed point satisfying (6), say $(\phi_0, \psi_0)$, the linearized system can be written as

$$\dot{x}_{j+1} = A_{11}x_j + A_{12}y_j \quad \text{and} \quad \dot{y}_{j+1} = A_{21}x_j + A_{22}y_j$$

where $x = \phi - \phi_0$, $y = \psi - \psi_0$, $A_{11} = 1 - a_2 \omega^2 \cos(\phi + \psi)$, $A_{12} = a_2 a_1 \omega^2 \sin(\phi + \psi)$, $A_{21} = a_2 a_1 \omega^2 \sin(\phi + \psi)$, $A_{22} = a_2 a_1 \omega^2 \sin(\phi + \psi)$.

Stability of the 2-periodic motion can be checked by examining the eigenvalues of the Jacobian matrix obtained from (7). In our simulations, the following parameters are chosen (same as in [2]) for the bouncing ball system $m = 1/26$, $e = 0.8$, $A = 0.013$ so that $a_1 = 1.73333$, $a_2 = -0.73333$, $\omega_1 = 0.0059342$ sec$^{-1}$. For the $n = 1$ and $m = 1$ case, the values of $\omega_n$ and $\omega_1$ are obtained as $19.0997$ rad/sec and 28.9505 rad/sec respectively. Hence, from the expression of $h$, the stable bounce heights for 1-periodic motion, range from 5.78 cm to 13.27 cm as $\omega_n$ is varied from $\omega_0$ to $\omega_1$. For $\omega_n > \omega_1$ 2-periodic fixed points ($n = 1$ and $m = 2$ case) are obtained. If, for example, $\omega_n = 29$ rad/sec is the operating frequency, the fixed points, in radians, obtained from (5) and (6) are $(\phi_1, \psi_1) = (5.7902, 6.5267)$, $(\phi_2, \psi_2) = (5.9527, 6.0396)$, and $(\phi_3, \psi_3) = (5.8346, 6.2831)$. It is clear from the value $\omega_3 = 2.8312\pi$, that the third fixed point corresponds to 1-periodic motion of the bouncing ball. Further, note that, $\psi_1 + \psi_2 = 2\pi$, implying that the first two solutions refer to a single pair of bounce heights in 2-periodic motion. That is, if the system (2) starts from the state $(\phi_1, \psi_1)$, then its successive states are $(\phi_2, \psi_2)$, $(\phi_3, \psi_3)$, $(\phi_4, \psi_4)$,...From the linearized equations it is observed that the fixed points corresponding to the 2-periodic motion are stable and the fixed point corresponding to 1-periodic solution is unstable for $\omega_1 < \omega_n < \omega_2$, where $\omega_2$ is the second flip-bifurcation value.

### 4 Frequency Modulated Plate Vibrations

In this section we investigate the effect of modulating the instantaneous frequency of the plate with a sinusoidal modulating signal [4]. In this method of modulation, the amplitude of the modulated wave remains a constant. Consider a sinusoidal modulating wave defined by $m(t) = A_m \cos(\omega_m t)$. The instantaneous frequency of the plate vibration with the above modulating signal is given by $\omega(t) = \omega_n + k_m A_m \cos(\omega_m t)$ where $\Delta \omega = \omega_m - \omega_n$ and $\omega_m$ is the frequency of the unmodulated plate. The quantity $\Delta \omega$ is called the frequency deviation, representing the maximum departure of the instantaneous frequency of the FM wave from the frequency $\omega_n$ of the unmodulated wave. The angle $\theta(t)$ of the FM wave is obtained as $\theta(t) = \int_0^t \omega_n(\tau) d\tau = \omega_n t + \frac{\pi}{\omega_m} \sin(\omega_m t)$. The ratio of the frequency deviation $\Delta \omega$ to the modulation frequency $\omega_m$ is commonly referred to as the modulation index of the FM wave. We denote it by $\beta$, so that we may write $\beta = \frac{\Delta \omega}{\omega_m}$, $\beta(t) = \omega_n t + \pi \sin(\omega_m t)$. Thus the frequency modulated plane motion is given by $y(t) = A \cos(\omega_n t + \beta \sin(\omega_m t))$.

![Figure 1: (a) 2-periodic motion of bouncing ball. (b) Plate Motion: $\beta = 0.6$ for the modulated wave.](image-url)
angle $\theta(t)$ from the angle $\omega_c t$ of the unmodulated wave. Depending on the value of the modulation index $\beta$, we may distinguish two cases of frequency modulation: (1) narrow-band FM for which $\beta$ is small, and (2) wide-band FM for which $\beta$ is large. We further assume that $\omega_m = R$, where $R$ is a rational number. This assumption is essential, because otherwise it is not possible to obtain periodic motion of the plate, and therefore fixed points for the motion of the ball. The modified state space equations are written as:

$$
\dot{\phi}_j = \phi_{j+1} + \psi_j, \\
\dot{\psi}_j = -a_1 \psi_j - a_2 \omega^2 \sin((\phi_j + \psi_j) + \beta \sin(\phi_j + \psi_j)) R (1 + \beta \cos(\phi_j + \psi_j))
$$

For 2-periodic motion it can be shown that $R$ must be of the form $\frac{2k}{\pi}$, where $k \in \mathbb{N}$, the set of natural numbers, because there must be integer cycles of the modulating frequency ($\omega_m$) within 2 cycles of the plate carrier frequency $\omega_c$. In our simulations, we have considered the case $R = 1$, i.e., the modulation frequency ($\omega_m$) is the same as the frequency of the unmodulated wave ($\omega_c$).

4.1 The Effect of $\beta$

The fixed points $(\phi_x, \psi_x)$ of (8), corresponding to 1-periodic motion of the ball must satisfy the same set of conditions $\phi_{j+1} = \phi_j$ and $\psi_{j+1} = \psi_j$ as in the unmodulated case. From (8) we can infer that $\psi_x = 2\pi x$, as before, with the bounce-height given by the expression for $h$. It follows therefore, that given a particular $\omega_c$, it is not possible to change the bounce-height of the ball by frequency modulation. $\phi_x$ however, changes, and can be obtained by solving numerically

$$
C = \sin(\phi_x + 2\pi \beta \sin(\phi_x))(1 + \beta \cos(\phi_x)),
$$

where $C = \frac{2\pi x}{\omega_c^2}$. In our simulations, with $\omega_c = 2\pi$, various values of $\phi_x$ are obtained by varying $\beta$, where each $\phi_x$ is obtained by solving (9) using the Newton-Raphson Method. Fig 2a graphs the variation of $\phi_x$ with $\beta$. At $\beta = 0$ change of stability of 1-periodic orbit occurs. Application possible to delay or hasten the first flip bifurcation by changing $\beta$ appropriately. If we linearize the map for 2-periodic FM motion about its fixed points, the entries of the Jacobian will be continuous functions of the parameter $\beta$. Since the eigenvalues of a square matrix are continuous function of its entries, the eigenvalues of the linearized system with frequency modulated motion will vary continuously with $\beta$. Hence the stability of the periodic orbits with sufficiently small $\beta$ is the same as that with $\beta = 0$. The fixed points associated with frequency modulated plate motion also changes continuously with $\beta$. Since both $\phi$ & $\psi$ change for FM 2-periodic motions, different bounce-heights can be obtained for the same $\omega_c$. Table 1 tabulates the stable and unstable fixed points at $\omega_c = 29$ for positive values of $\beta$. Fig 2b presents the corresponding bifurcation diagram. The top and the bottom curves correspond to stable 2-periodic motions, while the middle one refers to an unstable 1-periodic solution.

5 Conclusions

We have studied the behavior of a bouncing ball on a vibrating plate with special emphasis on multiple periodic motions. Frequency modulation on the sinusoidal plate motion has been introduced to control the ball motion so that arbitrary heights, within a certain range, for 2-periodic motion is achieved for a given frequency of the plate. The behavior of the modulation index $\beta$ with respect to the bifurcation values, has been studied by means of rigorous computer simulations. We have observed that bifurcation can be delayed by using frequency modulation. In future it will be interesting to investigate the possibility of controlling the onset of chaos in bouncing ball and similar other dynamical systems using frequency modulation.

References


