Pseudo-Inverse Based Iterative Learning Control for Nonlinear Plants with Disturbances.\textsuperscript{1}

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Abstract

Learning control is a very effective approach for tracking control in processes occurring repetitively over a fixed interval of time. In this paper, the stable-inversion based learning controller as presented in [1] is extended to accomodate a general class of nonlinear, nonminimum phase plants with disturbances and initialization errors. The extension requires the computation of an approximate inverse of the linearized plant rather than the exact inverse. An advantage of this approach is that the output of the plant need not be differentiatated. A bound on the asymptotic trajectory error is exhibited via a concise proof, and is shown to grow continuously with a bound on the disturbances. Simulation studies confirm that in the presence of bounded disturbances, the tracking error converges to a neighborhood of zero. The structure of the controller is such that the low frequency components of the trajectory converge faster than the high frequency components.

1 Introduction

Iterative learning control (ILC) refers to a class of self-tuning controllers where the system performance of a specified task is gradually improved or perfected based on the previous performances of identical tasks. ILC can also be used effectively when the plant cannot be modeled accurately. The most commonly seen applications of learning control are in the area of robot control in production industries, where a robot is required to perform a single task, say pick-and-place an object along a given trajectory, repetitively. With a feedback controller alone, the same tracking error would persist in every repeated trial. In contrast, a learning controller can use the information from the previous execution to improve the tracking performance in the next execution. While in some applications, the need to repeat a trajectory multiple times for learning may be a disadvantage, we focus our attention on those others where learning control is a natural solution.

For more than a decade researchers have defined and analysed learning control systems. First introduced by Arimoto et al. [2] and Craig [3] and later modified by many others including [4], [5], [6], ILC schemes strive to improve the performance of repetitive tasks using the information of the previous trial of the same task. Gao and Chen [7] have illustrated with counter examples the limitations of some of these learning algorithms with regard to nonminimum phase systems.

To remove the minimum phase requirement, they developed a learning algorithm for linear systems based on "stable-inversion". Based on the algorithm of Gao and Chen, we developed an iterative learning control algorithm in [1] for nonlinear nonminimum phase plants with input disturbances.

In this paper we propose a modification of the iterative learning control algorithm presented in [1] so that it can be applied to a more generic class of nonlinear nonminimum phase plants with input disturbance and output sensor noise. In section 2, a learning controller is proposed by formulating a pseudo-inverse of the linearized plant at the origin. A proof of convergence of the input trajectory to a neighborhood of the desired one is provided. In section 3, simulation examples are presented to show the performance of the proposed learning controller. Finally, section 4 concludes the paper.

1.1 Notation

\[ \| \cdot \| : \text{Euclidean norm.} \]
\[ \| f \|_{\infty, [0,T]} : \sup_{t \in [0,T]} \| f(t) \| \]
\[ L_\infty : \text{Functions such that } \| f \|_\infty < \infty \text{ on } (-\infty, \infty). \]
\[ L_\infty [0,T] : \text{Functions such that } \| f \|_{\infty, [0,T]} < \infty \text{ on } [0,T]. \]
\[ C^r [0,T] : \text{Continuous functions on } [0,T]. \]
\[ C^r [0,T] : \text{r times continuously differentiable functions on } [0,T]. \]
\[ B_r : \{ u(\cdot) \mid u(t) \in \mathbb{R}^n \text{ and } \| u(\cdot) \|_\infty < r < \infty \}. \]
\[ o(h) : \text{A function satisfying } \lim_{\| h \| \to 0} \frac{o(h)}{\| h \|} = 0. \]
\[ T : \text{T}(x(t)) = x(t) \forall t \in [0,T], \text{ and } \text{T}(x(t)) = 0 \text{ otherwise.} \]
\[ \mathcal{F} : \text{Fourier Transform mapping } \mathcal{F} : L_2 \to L_2. \]
2 Nonlinear Nonminimum Phase Plant with Disturbances

In this section we present a robust iterative learning algorithm for nonlinear systems. We consider only square (same number of inputs and outputs) time-invariant nonlinear systems.

2.1 System Description

Consider a nonlinear system which is stable-in-first-approximation at \( x = 0 \) and also input-to-state stable:

\[
\begin{align*}
x_i(t) &= f(x_i(t)) + g(x_i(t))u_i(t), \quad x_i(\pm \infty) = 0 \\
y_i(t) &= h(x_i(t));
\end{align*}
\]

(1)

where \( i \) is the index of iteration of ILC, \( \{u_i\}_{i=0}^\infty \) is a family of input sequence, \( x_i(t) \in \mathbb{R}^n, \ u_i(t) \) and \( y_i(t) \in \mathbb{R}^m \) and \( f: \mathbb{R}^n \to \mathbb{R}^n, \ g: \mathbb{R}^n \to \mathbb{R}^{n \times m}, \ h: \mathbb{R}^n \to \mathbb{R}^m \).

A desired trajectory \( y_d(t) \) is supported on finite interval \( [0, T] \) of the time axis. The objective of learning is to construct a sequence of input trajectories \( \{u_i\}_{i=1}^\infty \) such that \( u_i \to u^* \) and \( u^*(t) \) causes the system to track a trajectory \( y_d(t) \) as closely as possible on \([0, T]\)

\[
\begin{align*}
x_d(t) &= f(x_d(t)) + g(x_d(t))u_d(t), \quad x_d(0) = 0 \\
y_d(t) &= h(x_d(t));
\end{align*}
\]

(2)

is satisfied \( \forall t \in [0, T] \). We make the following assumptions:

(A1) The functions \( f(\cdot), \ g(\cdot), \ h(\cdot) \) are continuously differentiable and \( b \) is continuous.

(A2) \( u_i \in L_{\infty} \cap C^0 \cap B_r \).

(A3) The system is stable-in-first-approximation and input-to-state stable.

(Note: If the system is not stable it may be stabilized prior to application of our methods).

To model input and output disturbances, the plant equation (1) is modified:

\[
\begin{align*}
\dot{x}_i(t) &= f(x_i(t)) + g(x_i(t))u_i(t) + b(x_i(t))w_i(t), \\
y_i(t) &= h(x_i(t)) + v_i(t), \quad x_i(0) = 0;
\end{align*}
\]

(3)

where \( b: \mathbb{R}^n \to \mathbb{R}^{n \times \nu}, \ w_i \in \mathbb{R}^\nu, \ v_i \in \mathbb{R}^m \). The function \( w_i(t) \) represents both repulsive and random bounded disturbances of the system; it may be stiction, nonreproducible friction, state-independent modeling errors, etc. \( v_i(t) \) represents sensor noise. In addition, we also assume that:

(A4) The disturbances \( w_i(\cdot) \) and \( v_i(\cdot) \) are bounded by \( b_w \) and \( b_v \) respectively (i.e. \( \|w_i(t)\| \leq b_w \) and \( \|v_i(t)\| \leq b_v \).

For such a system an ILC is proposed as shown in Fig. 1.

2.2 Formulation of Learning Controller

In this section a good candidate for the learning controller \( LC \) of Fig. 1 is derived by first linearizing the plant. Since the nonlinear system (3) is input-to-state-stable (A5), and \( h \) is continuous (A1), it defines a causal nonlinear input-to-output map \( P \) as:

\[ P: u_i \to y_i; \quad L_{\infty}[0, \infty) \to L_{\infty}[0, \infty). \]

Since \( P \) is stable-in-first-approximation (A5), we define a stable time-invariant input-to-output linear operator \( DP[0] \) by linearizing the system (3) around \( x_i = 0, u_i = 0, v_i = 0 \) as follows:

\[
\begin{align*}
\delta\dot{x}(t) &= A\delta x(t) + B\delta u(t), \quad \delta x(0) = 0 \\
\delta y(t) &= C\delta x(t).
\end{align*}
\]

(4)

where \( A \triangleq f_x(0), \ B \triangleq g(0), \ C \triangleq h_x(0) \). Hence,

\[ DP[0]: \delta u \to \delta y; \quad L_{\infty}[0, \infty) \to L_{\infty}[0, \infty). \]

Since \( \delta u \in L_{\infty}[0, T] \) and \( A \) is Hurwitz (in (4)) we can replace \( \delta x(0) = 0 \) with \( \delta x(\pm \infty) = 0 \) and not alter the I-O map defined by (4) and hence the only map provided is \( 1 \). Consider the I-O map for the adjoint system:

\[
\begin{align*}
\delta\dot{z} &= -A^T\delta z - C^T\delta u, \quad \delta z(\pm \infty) = 0 \\
\delta\dot{y} &= B^T\delta z.
\end{align*}
\]

(5)

Since \( A \) is Hurwitz, \( -A^T \) is hyperbolic and has all eigenvalues on the right-half plane. Further, (5) defines a unique noncausal mapping as shown by Devasia et al. [8].

\[ DP[0]^*: \delta u \to \delta y; \quad L_{\infty} \to L_{\infty}. \]

The adjoint system satisfies the property \( \langle DP[0]u, v \rangle = \langle u, DP[0]^*v \rangle \) [9].

Denoting \( \delta x_i = x_i - x_i, \ \delta y_i = y_i - y_i, \ \delta u_i = u_i - u_i \), we derive (by Taylor series expansion) a linearized plant from (2) as follows:

\[
\begin{align*}
\dot{x}_i(t) + \delta\dot{x}_i(t) &= f(x_i + \delta x_i) + g(x_i + \delta x_i)[u_i + \delta u_i]; \\
\approx f(x_i) + f_x(x_i)\delta x_i + [g(x_i) + g_x(x_i)\delta x_i][u_i + \delta u_i]; \\
y_i(t) + \delta y_i(t) &= h(x_i + \delta x_i) + v_i.
\end{align*}
\]

(6)

where \( f_x(x_i(t)) \triangleq \frac{df_x}{dx}(x_i(t)); \ g_x(x_i(t)) \triangleq \frac{dg_x}{dx}(x_i(t)). \)

Subtracting (3) from (6) and neglecting higher order terms we obtain a linearized plant around the solution \( x_i(t) \) to (3):

\[
\begin{align*}
\delta\dot{x}_i(t) &= f_x(x_i)\delta x_i + g(x_i)\delta u_i + g_x(x_i)\delta x_iu_i - b(x_i)w_i; \\
\delta\dot{y}_i(t) &= h_x(x_i(t))\delta x_i(t) - v_i(t); \\
\delta x_i(0) &= 0
\end{align*}
\]

(7)
Since (4) is stable, it can be proved by Lyapunov methods that (7) is also BIBO stable if \( x_i \) lies within a certain bound. Note that, here also we can replace \( \delta x_i(0) = 0 \) (as in 4) with \( \delta x_i(\pm \infty) = 0 \) and not alter the I-O map. Define \( A_i(t) \Delta f_x(x_i(t)) + g_x(x_i(t))u_i(t) \), \( B_i(t) \Delta g(x_i(t)) \), \( C_i(t) \Delta h_u(x_i(t)) \), \( b_i(t) \Delta b(x_i(t)) \). The stable linear system (7) has a solution and defines a linear I-O map:

\[
DP_{|u_i|} : \delta u_i \to \delta y_i; \quad L_\infty \to L_\infty.
\]

**Defining \( DP_{|u_i|}^\alpha \)**

Motivated by the concept of a pseudo-inverse [10] we define learning controller by the following linear operator:

\[
DP_{|u_i|}^\alpha \Delta (\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1}DP_{|u_i|}^*
\]  

(8)

for \( \alpha \neq 0 \) We call this “approximate inverse” the \( \alpha \)-pseudo inverse of \( DP_{|u_i|} \). For simplicity the \( \alpha \)-pseudo inverse is referred to as simply a pseudo-inverse in the rest of this paper. In time-domain using (4) and (5) \((\alpha I + DP_{|u_i|}^*DP_{|u_i|}) : \delta u \to \delta y \) is:

\[
\begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
-C^T C & -A^T
\end{bmatrix} \begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} + \begin{bmatrix}
B \\
0
\end{bmatrix} \delta u;
\]

\[
\delta y = a \delta u + \delta \gamma = a \delta u + B^T \delta \tilde{z}; \quad \begin{bmatrix}
\delta \tilde{x}(\pm \infty) \\
\delta \tilde{z}(\pm \infty)
\end{bmatrix} = 0
\]

(9)

Since \( DP_{|u_i|} \) is stable, the above system (9) is hyperbolic with eigenvalues \( \lambda(A) \cup \lambda(-A^T) \). Hence, by [8] \((\alpha I + DP_{|u_i|}^*DP_{|u_i|}) : \delta u \to \delta \gamma; \quad L_\infty \to L_\infty \) and is noncausal. Solving (9) for \( \delta u \), we see that the inverted operator \((\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1} \) is:

\[
\begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} = \begin{bmatrix}
A & -Ba^{-1}B^T \\
-C^T C & -A^T
\end{bmatrix} \begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} + \begin{bmatrix}
0 \\
Ba^{-1}
\end{bmatrix} \delta y
\]

\[
\delta u = a^{-1}(\delta \gamma - B^T \delta \tilde{z}); \quad \begin{bmatrix}
\delta \tilde{x}(\pm \infty) \\
\delta \tilde{z}(\pm \infty)
\end{bmatrix} = 0
\]

(10)

The eigenvalues of the above system are continuous functions of \( \alpha \). In the limit \( \alpha \to \infty \), \( A_{\alpha} \) is hyperbolic (since \( A \) is Hurwitz). Thus we can always choose an \( \alpha \) for which \( A_{\alpha} \) is hyperbolic. The system (10) is solved by the stable-noncausal-solution approach of Devasia et al.[8]. Hence, \((\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1} : \delta \gamma \to \delta u; \quad L_\infty \to L_\infty \).

The learning controller \((\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1}DP_{|u_i|}^* \) is given in time-domain by:

\[
\begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} = \begin{bmatrix}
A & -Ba^{-1}B^T \\
-C^T C & -A^T
\end{bmatrix} \begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} \delta y; \quad X(\pm \infty) = 0
\]

\[
\begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix} = \begin{bmatrix}
0 & -a^{-1}B^T \\
0 & a^{-1}B^T
\end{bmatrix} \begin{bmatrix}
\delta \tilde{x} \\
\delta \tilde{z}
\end{bmatrix}
\]

(11)

\( A_{\alpha} \) is block diagonal, therefore the eigenvalues of \( A_{\alpha} \) are eigenvalues of \( A_{\alpha} \) (equation (10)) and \(-A^T\). Since \( A_{\alpha} \) is hyperbolic for some \( \alpha \), \( A_{\alpha} \) is hyperbolic. Hence, \( \delta y \to \delta u \) \( L_\infty \to L_\infty \)

\[
(\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1}DP_{|u_i|}^* : \delta y \to \delta u
\]

and the solution of the linear controller described by (11) can be obtained using stable-noncausal-solution approach [8]. (Using initial conditions at \( t = -\infty \) rather than \( t = 0 \) allows us to control \( X(0) \) via \( \delta y(-\infty, 0) \). Thus tracking performance can be improved relative to assumptions of \( \delta y \equiv 0 \) on \((-\infty, 0) \) and \( X(0) = 0 \). Now the update law of the ILC is written in terms of the operators \( P_i = (\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1}DP_{|u_i|}^* \) and \( T \) for \( t \in [0, T] \) is:

\[
u_{i+1}(t) = T(u_i + \delta u_i)
\]

\[
= T(u_i + (\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1}DP_{|u_i|}^*(y_i - P(u_i)))
\]

\[
= T(u_i + (\alpha I + DP_{|u_i|}^*DP_{|u_i|})^{-1}DP_{|u_i|}^*(y_i - y_i))
\]

(12)

Note that if \( u_0 = T(u_0), T(u_i(t) + \delta u_i(t)) = u_i(t) + T(\delta u_i(t)) \) for all \( i \). (Note that in Figure 1 the truncation operator \( T \) is placed before the summation junction).

### 2.3 Convergence Analysis

**Definition 1** We define \( \lambda \) norm for a function \( x : [0, T] \to \mathbb{R}^n \) by

\[
||x(\cdot)||_\lambda \Delta \sup_{t \in [0, T]} e^{-\lambda t}||x(t)||
\]

(13)

Note that \( ||x||_\lambda \leq ||x||_\infty < e^{\lambda T}||x||_\lambda \) for \( \lambda > 0 \), implying that \( ||x||_\lambda \) and \( ||x||_\infty \) are equivalent norms. Thus convergence results can be proved using either norm. 

**Induced \( \lambda \)-norm:** \( ||A||_\lambda = \sup_{||u||_1 = 1, ||u||_{\infty} = 0} ||Au||_\lambda \)

**Condition 1** Define the Fourier Transform of \( DP_{|u_i|} \) by \( \mathcal{F}(DP_{|u_i|}) = \hat{DP}_{|u_i|}(f) \). \( ||\hat{DP}_{|u_i|}(f)|| > \beta > 0 \) \( \forall f \) (i.e no finite or infinite zeros on \( j\omega \) axis).

Note that \( L_\infty[0, T] \subset L_2 \).

**Theorem 1** If the assumptions (A1-A5) imposed above hold and Condition 1 is satisfied, then the algorithm (12) produces a sequence of inputs which converges to \( u^* \) if there are no disturbances (i.e. \( u_t = 0 \) and \( v_t = 0 \)) and no initialization errors. If \( u_t \) and \( v_t \) are bounded, \( u_t \) converges to a ball \( B(u^*, \beta) \), as \( s \to \infty \).
The radius $r$ of the ball $B(u^*, r)$ depends continuously on the bounds on the disturbances $w_1$ and $v_1$. If there exists a $u_0 \in L_0 \cap C_0[0, T]$ with $P(u_0) = y_0$, then $u_t$ converges to the desired input solution $u_d$.

**Proof:** The proof relies on the application of a variant of the contraction mapping theorem [11] to the input sequence. The main idea of the proof is to show that $\|\delta u_{t+1}\| \leq p\|\delta u_t\| + b_d$ where $0 \leq p < 1$ ($\delta u_t \equiv u_{t+d} - u_t$). This implies that $\lim_{t \to \infty} \sup \|\delta u_t\| \to \frac{1}{1-p}b_d$, where $b_d$ is a continuous function of the bounds on disturbances and initialization error.

Construct the sequence $(u_i(\cdot))_{i=0}^\infty$ by defining:

$$ u_0 = T(u_0), \quad (i.e. \ u_0 \ has\ compact\ support) $$

$$ u_{i+1} = T_i(u_i(\cdot), u_{i+1}(\cdot)) $$

$$ \Delta T_i(u_i(\cdot), w_i(\cdot)) [u_i(\cdot)] \text{ is denoted by } T_i(u_i) \text{ for simplicity in the rest of this paper. Now from (12) and linearity of the truncation operator:}$$

$$ \|T_i(u_i) - T_i(v_i)\| = \|T(u_i + (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})(y_0 - P(u_i))) - T(v_i + (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})(y_0 - P(v_i)))\| $n

$$ = \|T(u_i - (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})(y_0 - P(u_i))) - T(v_i - (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})(y_0 - P(v_i)))\| $n

$$ = \|T(u_i - (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})P(u_i) - v_i + (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})P(u_i + v_i - w_i))\| $n

Paden and Chen [12] show that the Fréchet derivative of $P$ is given by $DP_t(u)$, (see Appendix [12]). That is:

$$ \lim_{\|\delta u_t\| \to 0} \frac{\|P(u_t + \delta u_t) - P(u_t) - DP_t(u_t)[\delta u_t]\|}{\|\delta u_t\|} = 0 \quad (16) $$

In (16) let $s(\delta u_t)$ be defined by: $s(\delta u_t) \equiv P(u_t + \delta u_t) - P(u_t) - DP_t(u_t)[\delta u_t]$. From (16) we can see $s$ is $o(\delta u_t)$. Denoting $\delta u_t \equiv (\delta u_t - v_t)$ we can rewrite (16) as:

$$ \|T_i(u_i) - T_i(v_i)\| = \|T(-\delta u_i + (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1}P(u_i) - (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1}P(u_i + v_i - w_i))\| $n

$$ = \|T(-\delta u_i + (aI + DP_iDP_i^{-1}DP_0^{-1}DP_0^{-1})(s(\delta u_i) + P(u_i + v_i - w_i))\| $n

$$ + \|DP_t(u_t)[\delta u_t]\| \|s(\delta u_t)\| $$

(17)

Since $s(\delta u_t)$ is $o(\delta u_t)$,

$$ \lim_{\|\delta u_t\| \to 0} \frac{\|\delta u_t\|}{\|\delta u_t\|} = 0. \quad (18) $$

This implies that $\forall \epsilon > 0, \exists \delta > 0$ such that $\|\delta u_t\| \leq \delta$.

$$ \frac{\|\delta u_t\|}{\|\delta u_t\|} \leq \epsilon < 1 \quad (19) $$

**Bounding $\delta z_1$ and $\delta x$**

From assumptions A1, A3, A4: $\|z_1\| \leq \|z_1(x_1)\| + \|z_1(x_1)\| \leq b_i$, $\|\delta x_1\| \leq b_z_1$, $\|\delta x_1\| \leq b_0$. Therefore, $\|A_i\| \leq b_1$, $\|B_i\| \leq b_2$, $\|C_i\| \leq b_3$. From (7) we can write:

$$ \delta z_1(t) = \delta x_1(0) + \int_0^t A_i(t)\delta x_1(t) + B_i(t)\delta u_i(t) \quad (20) $$

Therefore, using the triangle-inequality and bounds on $z_1$ and $u_i$ we have,

$$ \|\delta z_1\| \leq \|\delta z_1(0)\| + \int_0^t \|A_i(t)\|\|\delta x_1(t)\| $$

$$ + \|B_i(t)\|\|\delta u_i(t)\| + b_b\|\delta z_1\| \quad (21) $$

Using Gronwall-Bellman inequality (see [11], p. 63)

$$ \|\delta z_1\| \leq e^{\lambda t}\|\delta z_1(0)\| + \int_0^t e^{\lambda(t-r)}(b_b\|\delta u_i(t)\| $$

$$ + b_b\|\delta z_1\|) \quad (22) $$

Multiplying (21) by $e^{-\lambda t}$ defining $K_1 \equiv \max(b_1, b_2)$ and assuming $\lambda > K_1$ we have

$$ e^{-\lambda t}\|\delta z_1\| \leq e^{(b_1-\lambda)x}(\|\delta z_1(0)\| + K_1\int_0^t e^{(K_1-\lambda)(t-r)} $$

$$ e^{-\lambda t}\|\delta u_i(t)\|dr + b_b\int_0^t e^{-\lambda(t-r)}\|\delta z_1\| \quad (23) $$

Note that for a constant $\|k\|_1 = k$. Taking sup over $t \in [0, T]$ we have

$$ \|\delta z_1\|_1 \leq \|\delta z_1(0)\| + \frac{K_1}{\lambda - K_1} (1 - e^{(K_1-\lambda)T}) $$

$$ \|\delta u_i\|_1 + \frac{b_b}{\lambda - K_1}(1 - e^{(b_1-\lambda)T}) \quad (22) $$

Similarly from (4) it can be proved:

$$ \|\delta z_1\|_1 \leq \|\delta z_1(0)\| + \frac{K_1}{\lambda - K_1}(1 - e^{(K_1-\lambda)T})\|\delta u_i(t)\|_1 \quad (23) $$

where $\delta u_i$ is the input to (4).

**Defining $\Delta DP_{u_t}$**

Define a linear operator $\Delta DP_{u_t}: \delta u_t \mapsto \Delta y_t; \quad L_0 \to L_\infty$. (24)

From (7) the output of the operator $DP_{u_t}$ is: $\delta y_t(t) = C(t)\delta x_1(t) - v_i(t)$ and (4) the output of the operator $DP_0$ is $\delta y_t(t) = C_0\delta x_1(t)$ (this implies, $\delta y_t(t) = C(t)\delta x_1(t) - v_i(t) - C_0\delta x_1(t)$). Therefore, using equations (22), (23) and bound on $v_i$ we can write,

$$ \|\Delta y_t\|_1 \leq \|C_0\|_1\|\delta x_1\|_1 + \|C_i\|_1\|\delta x_1\|_1 + b_v $$

$$ \leq \frac{2b_z_1K_1}{\lambda - K_1} (1 - e^{(K_1-\lambda)T})\|\delta u_i\|_1 + b_v + \frac{b_z_1b_b}{\lambda - K_1} (1 - e^{(b_1-\lambda)T}) $$

$$ + b_z_1(\|\delta x_1(0)\|_1 + \|\delta x(0)\|_1) \quad (25) $$
Showing Contraction Mapping
From (17) we have,
\[ ||T_i(u_l) - T_i(v_l)|| = \| T_i(\delta u_i + (\alpha I + DP)^{-1}(\alpha I + DP)\alpha I)|_{i=0}^{\alpha I} \| \]
\[ \leq \| T_i(\delta u_i + (\alpha I + DP)^{-1}(\alpha I + DP)\alpha I) \| + \epsilon\| T_i(\delta u_i) \| \] \text{from (19)}
\text{substituting } DP|_{i=0}^{\alpha I} = DP|_{i=0}^{\alpha I} + \Delta DP|_{i=0}^{\alpha I}\]
\[ = \| T_i(\delta u_i + (\alpha I + DP)^{-1}(\alpha I + DP)\alpha I) \| + \epsilon\| T_i(\delta u_i) \| \]
\[ = \| T_i(\delta u_i + (\alpha I + DP)^{-1}(\alpha I + DP)\alpha I) \| + \epsilon\| T_i(\delta u_i) \| \]
\[ = \| T_i(\delta u_i + (\alpha I + DP)^{-1}(\alpha I + DP)\alpha I) \| + \epsilon\| T_i(\delta u_i) \| \]
\[ \leq \| T_i(\delta u_i + (\alpha I + DP)^{-1}(\alpha I + DP)\alpha I) \| + \epsilon\| T_i(\delta u_i) \| \]
(26)

Define \( \|(\alpha I + DP)^{-1}(\alpha I + DP)\alpha I\| \leq \gamma_1 \). It can be shown in the following way that if \( DP|_{i=0}^{\alpha I} \) satisfies \textbf{Condition 1} then \( \|(\alpha I + DP)^{-1}(\alpha I + DP)\alpha I\| \leq \epsilon_0 \|\delta u_i\|\|\lambda \),
where \( 0 < \epsilon_0 < 1 \). With the choice of \( \alpha \) sufficiently small, \( \epsilon_0 \) can be made arbitrarily small.

Let \( \tilde{y} = DP|_{i=0}^{\alpha I} \tilde{u}, \tilde{y}(t) := \tilde{F}(\tilde{u}(t)) \)
\[ \| DP|_{i=0}^{\alpha I} \tilde{u} \|^2 \leq \int_{-\infty}^{\infty} \tilde{y}(t) \tilde{y}(t)^T dt = \int_{-\infty}^{\infty} \tilde{F}(\tilde{u}(t)) \tilde{F}(\tilde{u}(t))^T dt \]
If \textbf{Condition 1} is satisfied, then
\[ \| DP|_{i=0}^{\alpha I} \tilde{u} \|^2 > \beta^2 \|\tilde{u}\| \] where \( \beta > 0 \).
(27)

Consider again equation (26). Let \( \delta u_i = (DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\tilde{u} \) so that \( \|(DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\delta u_i \| = \|\tilde{u}\|^2 \). Note that
\[ \|(DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\tilde{u} \|^2 \]
\[ \leq \|(DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I})\tilde{u} \|^2 + 2\alpha \langle DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} \tilde{u}, \tilde{u} \rangle + \alpha^2 \|\tilde{u}\|^2 \]
\[ > (2\alpha^2 \|\tilde{u}\|^2 + \alpha^2 \|\tilde{u}\|^2) \]
\[ \|\tilde{u}\|^2 < (2\alpha^2 + 2\alpha^2 - 1) \|(DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\tilde{u} \|^2 \]
(28)

Therefore, we can write,
\[ \|\alpha (DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\delta u_i \| = \alpha^2 \|\tilde{u}\|^2 \]
using (28)
\[ \leq \alpha^2 (2\alpha^2 + \alpha^2 - 1) \|(DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\tilde{u} \|^2 \]
\[ = \alpha^2 \|\tilde{u}\|^2 \]
(29)
where \( \epsilon_0 = \alpha^2 (2\alpha^2 + \alpha^2 - 1)^{1/2} < 1 \). With the choice of \( \alpha, \epsilon_0 \) can be made arbitrarily small.

If the transfer function corresponding to \( DP|_{i=0}^{\alpha I} \) is strictly proper then the \textbf{Condition 1} is not satisfied at \( f = \infty \). Then as \( f \to \infty \), \( \epsilon_0(f) \to 1 \) and, intuitively, high frequency components of the input sequence would converge slowly. In that case, the learning controller must be modified in the following way:

\[ \text{Figure 2: Modified operator } \tilde{DP}_{\text{a}} \]

Instead of considering \( (DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\tilde{u} \) as the learning operator, take \( (DP|_{i=0}^{\alpha I} DP|_{i=0}^{\alpha I} + \alpha I)|\tilde{u} \) as the modified learning controller, where \( \tilde{DP}_{\text{a}} \) is obtained by adding a feedforward term to \( DP|_{i=0}^{\alpha I} \) as shown in Fig. \textit{(2)}. \( DP|_{i=0}^{\alpha I} \) is given by modified equation (4) as follows:
\[ \delta x(t) = A\delta x(t) + B\delta u(t), \quad \delta x(\pm \infty) = 0 \]
\[ \delta y(t) = C\delta x(t) + \epsilon_0 \delta u(t). \]
(30)
where \( 0 < \epsilon_0 < 1 \).

The modified operator satisfies \textbf{Condition 1} and the convergence analysis proceeds in the same way with \( \epsilon_0 \) sufficiently small. Substituting the bounds on \( \|\Delta y_i\| = \|\Delta u_i\| \|\delta u_i\| \) from equation (25) and multiplying (26) by \( e^{-\lambda t} \), we can write \( \lambda = \text{norm of (26)} \) taking sup over \( t \in [0, \infty] \) as:
\[ \|T_i(u^*) - T_i(v_l) \| \leq G_1 \|\Delta y_i\| \|\delta u_i\| \] \[ \leq G_1 \left( 2b_\lambda K_1 (1 - e^{(K_1 - \lambda)^T}) \|\delta u_i\| \right. \]
\[ + b_\lambda b_\lambda b_\lambda \right) \left. + b_\lambda \|\delta x(0)\| \lambda \right. \]
\[ + \|\delta x(0)\| \lambda \right. \] \[ + \|\delta x(0)\| \lambda \right. \] \[ + \|\delta u_i\| \right. \]
\[ = G_1 \left( 2b_\lambda K_1 (1 - e^{(K_1 - \lambda)^T}) \right. \]
\[ \left. + \|\delta u_i\| \right. \]
\[ \leq \rho \|\delta u_i\| \lambda + b_d. \]
(31)
where \( K_0 \) is the norm bounds of the initial state errors. \( K_w b_w \) and \( K_w b_w \) are the norm bounds of the input and output disturbances respectively. Since \( \epsilon < 1 \), with sufficiently small \( \alpha \), we can find a \( \lambda > K_2 \geq K_1 \) which makes \( \rho \leq \rho < 1 \). Therefore, we can write:
\[ \|v_{i+1} - v_i\| \|\delta u_i\| \] \[ \leq \rho \|v_{i+1} - v_i\| \lambda + b_d \]
\[ \leq \rho \|v_{i+1} - v_i\| \lambda + b_d \]
\[ \leq \rho \|v_{i+1} - v_i\| \lambda + b_d \]
where \( b_d \) combines the norm bounds of the initial state errors of the controller and disturbances. Therefore, \( \lim_{i \to \infty} \|v_{i+1} - v_i\| \lambda \leq \frac{1}{1 - \rho} b_d \), i.e., \( \exists N \) such that \( h_i > N, u_i \in B(u^*, r) \), where \( u^* \) is the fixed point of the contraction mapping \( T_i \) and \( B(u^*, r) \) is an open ball of...
radius \( r = \frac{1}{1-\rho} b_d \) and center \( u^* \). If the disturbances and initialization errors are absent, \( b_d = 0 \) and hence \( u_i \) converges to \( u^* \). If \( \exists u_d \) such that \( P(u_d) = y_d \), the fixed point \( u^* \) of the contraction mapping \( T_i(\cdot) \) is shown to be \( u_d \) in the absence of \( u_i, v_i \) and initialization error. If \( u_i = u_d, y_i = y_d \) and \( \delta y_i = y_d - y_i = 0 \). This implies the output \( (\delta u_i) \) of the learning controller \( (\alpha I + DP_i^0 + DP_i^0)D(P_i^0)^{-1}DP_i^0 \) is zero. Therefore, \( T_i(u_d) = u_d + T((\alpha I + DP_i^0)D(P_i^0)^{-1}DP_i^0(y_d - y_d))(u_d + 0) = u_d \). \( \Rightarrow u^* = u_d \).

3 Simulation Results

3.1 Simulation Results with Input Disturbances

![Figure 3](image)

Figure 3: Tracking of nonlinear nonminimum phase system with input disturbance (a) after 3 iterations, (b) after 10 iterations.

In this section we perform simulation studies with a SISO nonlinear nonminimum phase plant \( P \), stable-in-first-approximation and also input-to-state stable, with input disturbance described by:

\[
\begin{aligned}
\dot{x}_1(t) &= -x_1 + x_2 \\
\dot{x}_2(t) &= -3x_2 + x_1^2 \\
\dot{x}_3(t) &= x_1 - 2x_3 \\
\dot{x}_4(t) &= -x_4 + x_3^2 \\
\end{aligned}
\]

\[
\begin{bmatrix}
\begin{bmatrix}
  f(x) \\
  g(x) \\
  b(x)
\end{bmatrix}
\end{bmatrix} + \begin{bmatrix}
  0.2 + 0.1 \sin^2(x_4) \\
  0 \\
  0 \\
  0
\end{bmatrix} u_i
\]

\[
y_i(t) = \frac{x_1(t) - 3x_3(t) + v_i(t)}{h(x)}; \quad x(\pm \infty) = 0. \quad (32)
\]

First we consider the output disturbance \( v_i(t) \) to be absent. The reference output trajectory is given by:

\[
y_0(t) = \begin{cases}
0.2 \sin(t) & t \in [0, 2\pi] \\
0 & \text{otherwise}
\end{cases} \quad (33)
\]

\( DP_0^0 \) is defined by linearizing the system (32) as:

\[
\begin{bmatrix}
  -1 & 1 & 0 & 0 \\
  0 & -3 & 0 & 0 \\
  1 & 0 & -2 & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix} \delta x + \begin{bmatrix}
  0.2 \\
  0 \\
  0 \\
  0
\end{bmatrix} \delta u,
\]

\[
\begin{bmatrix}
  1 & 0 & -3 & 0
\end{bmatrix} \delta x(t); \quad \delta x(\pm \infty) = 0.
\]

Since the linear controller is unstable, we apply non-causal stable solution approach[8]. We introduce \( w_i \) as a bounded input disturbance. \( w_i \) is normally distributed random numbers bounded between \( \pm 1 \). Matlab simulation (Fig. 3(a), Fig. 3(b)) shows near perfect tracking of the desired output trajectory after a couple of iterations. Note that the remaining error resulting from slow convergence of high frequency components.

![Figure 4](image)

Figure 4: Tracking of nonlinear nonminimum phase system with input & output disturbances after 3 iterations.

3.2 Simulation Results with Input & Output Disturbances

Now we introduce \( v_i \) as a random bounded output disturbance to the same nonlinear system given by (32). Input disturbance \( w_i \) as introduced earlier is also present. Matlab simulation (Fig. 4) shows good tracking of the desired output trajectory after 3 iterations.

3.3 Discussion

This ILC scheme has some advantages over that presented in [1]. In [1] the inverse of the linearized plant, \( DP_0^{-1} \) is taken to be the learning operator. This necessitates taking the derivative of the output to invert the system. Sugie and Ono [13] demonstrated the need for differentiation in the learning operator of most of the existing ILC schemes for systems without direct feed-through terms in output equations of plants. In practice, derivatives cannot be reliably computed in the presence of output sensor noise. Furthermore, the plant may itself produce an output signal that is not differentiable. In this new learning algorithm, however, it is not necessary to take the derivative of the output in
order to calculate the update term of the system input at every iteration. (Note that $\alpha$ should be nonzero).

Figure 5: Frequency responses (a) Linearized plant $DP_0$ (b) Exact inverse $DP_0^{-1}$ & Pseudo inverse $DP_0^{\alpha}$

The frequency responses of the linearized plant $DP_0$ (as given by equation (4)) and its exact inverse $DP_0^{-1}$ and pseudo-inverse $DP_0^{\alpha}$ (with $\alpha = 0.001$) are shown in Fig. 5a and Fig. 5b. In our previous scheme [1], the learning operator $DP_0^{-1}$ has high gain at high frequency as shown in Fig. 5b. Therefore, the high frequency noise is amplified by the learning operator. The frequency response of the learning controller $DP_0^{\alpha}$ proposed in this paper, is shown in Fig. 5b (with $\alpha = 0.001$). From Fig. 5b we see that the frequency response of $DP_0^{\alpha}$ behaves similarly to $DP_0^{-1}$ at low frequencies, but rolls off at high frequencies demonstrating a lowpass nature. Thus the high frequency sensor noise is filtered out. The phase responses of the exact inverse and the pseudo-inverse are identical (see Fig. 5b). Note that $(\alpha I + DP_0^{-1})$ is a zero phase filter. Excellent tracking of the low frequency components is achieved after a few iterations, while the high frequency components of the output error signal converge more slowly. This behavior is corroborated by Fig. 3a and Fig. 3b, where we see that the low frequency error converges to zero within the first few iterations, while the high frequency error spikes take a larger number of iterations to decay.

The learning controller discussed in this paper can be applied to a generic class of nonminimum phase plants. However, the controller design requires the precise knowledge of the linearized plant. In [14] we present results that allow a pseudo-inverse based ILC for plants with uncertainties. A proof of convergence is presented along with the condition for convergence that quantify the perturbation allowable for the learning algorithm to converge. The performance of the approach is also illustrated with simulation results.

4 Conclusion

The learning algorithm presented in this paper guarantees learning, under quite general assumptions. Theoretical assertions are corroborated by simulation results which demonstrate that in the presence of random bounded disturbances the tracking error is uniformly bounded. A major advantage of this scheme is that we are able to eliminate the differentiation operator from the learning update law, which allows us to consider a more general class of nonlinear plants.

References


