ULL, November 11, 2004

On some problems from classical boundary layer theory

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Motivation

Concept of boundary layer (Prandtl, 1904):

- Great impact on aerodynamics (nature of drag, origin of circulation, flow separation)
- Method of singular perturbations (Kaplun, Lagerstrom)
- Thin layer approximation (parabolic type equations)

Consequences of approximation:

- New self-similar solutions? *e.g.* wall jets[a] (incompressible)
- Non-physical behavior? *e.g.* no upstream influence[b] (compressible)

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[a] Krechetnikov R.V., Lipatov I.I., SIAM J. Appl. Math. **62**, 1837 (2002)

[b] Krechetnikov R.V., Lipatov I.I., J. Appl. Mech. Tech. Phys. **40**, 461 (1999).

Wall jets: self-similarity of the second kind

Blasius boundary layer

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = \Psi_{yyy}$$

$$y = 0: \;\; \Psi = \Psi_y = 0$$

$$y = \infty : \Psi_y = 1$$

Self-similarity:

$$\Psi = x^{1/2} f(\eta), \;\; \eta = \frac{y}{x^{1/2}}$$

Free jet

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = \Psi_{yyy}$$

$$y = 0: \;\; \Psi = \Psi_{yy} = 0$$

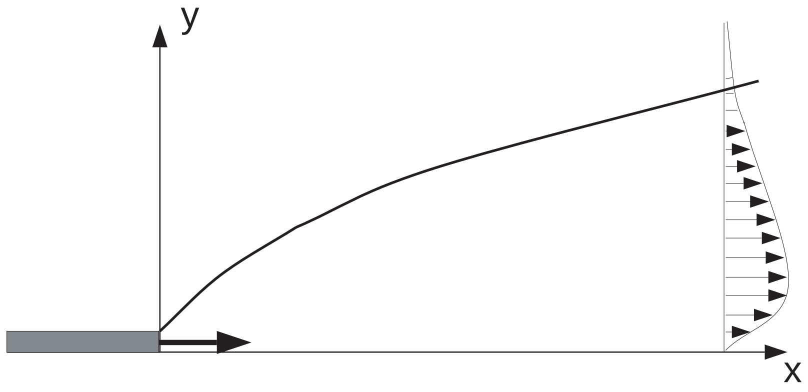
$$y = \infty : \Psi_y = 0$$

Self-similarity:

$$\Psi = x^{1-k} f(\eta), \;\; \eta = \frac{y}{x^k}, \;\; k =?$$

Momentum conservation, $\int\limits_{-\infty}^{+\infty} \Psi_y^2 \mathrm{d}y = \text{const}$, yields $k = \frac{2}{3}$

Wall jets: self-similarity of the second kind
(or problem of hidden invariances)



2D Wall jet

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = \Psi_{yyy}$$

$$y = 0: \ \Psi = \Psi_y = 0$$

$$y = \infty : \Psi_y = 0$$

Self-similarity:

$$\Psi = x^{1-k} f(\eta), \ \eta = \frac{y}{x^k}, \ k = ?$$

Akatnov (1953), Glauert (1956): $\frac{\partial}{\partial x} \left[\int\limits_0^{+\infty} u^2 \Psi \mathrm{d}y \right] = 0$ yields $k = \frac{3}{4}$.

Natural questions: is k unique? 3D case?

On generating functions method[a]

$S = (S_1, \ldots, S_n)$ is a conservation current for equation $\mathcal{Y} = \{F = 0\}$, $F = (F_1, \ldots, F_l)$ with n independent and m dependent variables, if

$$\sum_{i=1}^{n} D_i(S_i) = \sum_{j=1}^{l} A_j(F_j) = 0,\ A_j = \sum_{\sigma} a^j_\sigma D_\sigma,\ D_\sigma = \frac{\partial}{\partial x_\sigma} + u^i_\sigma \frac{\partial}{\partial u^i} + \cdots$$

Operators $A_j = \sum\limits_{\sigma} a^j_\sigma D_\sigma$, $j = 1, \ldots, l$ are determined from the solution of

$$l^*_F(\Omega) = 0,\ \Omega = \left(A^*_1(1), \ldots, A^*_l(1)\right)|_{\mathcal{Y}^\infty},$$

using definition of the formally adjoint operator $A^*_j = \sum\limits_{\sigma} (-1)^{|\sigma|} D_\sigma \circ a^j_\sigma$

l_F is a *universal linearization* operator:

$$l_F = \sum_{\sigma} \begin{pmatrix} \frac{\partial F_1}{\partial p^1_\sigma} & \ldots & \frac{\partial F_1}{\partial p^m_\sigma} \\ \ldots & \ldots & \ldots \\ \frac{\partial F_l}{\partial p^1_\sigma} & \ldots & \frac{\partial F_l}{\partial p^m_\sigma} \end{pmatrix} D_\sigma, \quad p^j_\sigma = \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}$$

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[a] *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, A. M. Vinogradov, AMS 1999

Solution for 2D wall jet

System $\left(l_F^{\epsilon}\right)^*(\Omega)=0$ *where* $\Omega=(\varphi_1,\varphi_2)^T$

$$u\frac{\partial\varphi_1}{\partial x}+v\frac{\partial\varphi_1}{\partial y}+\varphi_1\frac{\partial v}{\partial y}+\frac{\partial\varphi_2}{\partial x}+\nu\frac{\partial^2\varphi_1}{\partial y^2}=0\ \&\ \varphi_1\frac{\partial u}{\partial y}-\frac{\partial\varphi_2}{\partial y}=0,$$

or, upon elimination of φ_2,

$$\nu\frac{\partial^3\varphi}{\partial y^3}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial^2\Psi}{\partial x\partial y}\frac{\partial\varphi}{\partial y}+2\frac{\partial^2\Psi}{\partial y^2}\frac{\partial\varphi}{\partial x}+\frac{\partial\Psi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}=0.$$

The only dependence one can assume is $\varphi=\varphi(\Psi)$, which yields

$$\varphi_{\Psi\Psi}=0\ \Rightarrow\ \varphi=1\ \&\ \varphi=\Psi$$

The conserved current and hidden invariant

$$S=\begin{pmatrix} u^2\Psi \\ uv\Psi+\frac{u^2}{2}-\Psi\frac{\partial u}{\partial y}\end{pmatrix}\ \Rightarrow\ \frac{\partial}{\partial x}\left[\int_0^{+\infty}u^2\Psi dy\right]=0$$

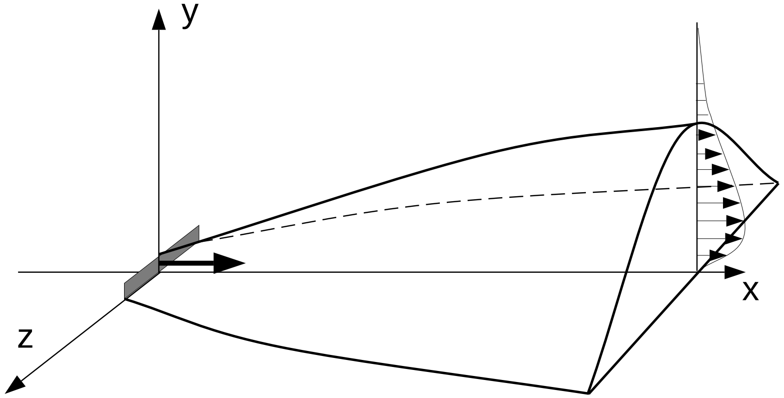
3D wall jet of Prandtl BLEs

Boundary layer equations:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\nu\frac{\partial^2 u}{\partial y^2},$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\nu\frac{\partial^2 w}{\partial y^2},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$



$$y=0:\ \ u=v=w=0,\ \ y=\infty:\ \ u=w=0,$$

$$z=0:\ \ \frac{\partial u}{\partial z}=\frac{\partial w}{\partial z}=0,\ \ z=\pm\infty:\ \ u=w=0.$$

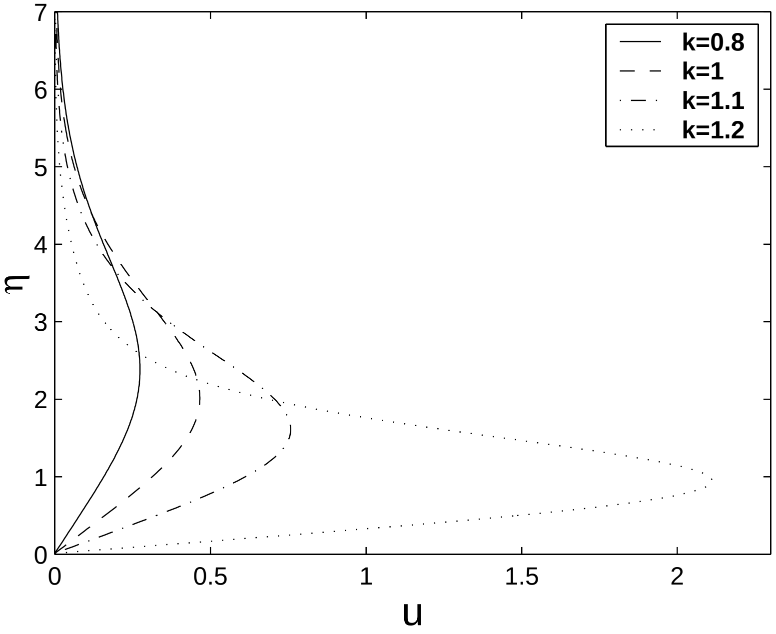
Self-similarity ($u=\Psi^1_y$, $w=\Psi^2_y$, $v=-\Psi^1_x-\Psi^2_z$):

$$\Psi^1=\nu x^{1-k}f\left(\eta,\zeta\right),\ \ \Psi^2=\nu x^{l-k}\varphi\left(\eta,\zeta\right),\quad \text{with } \eta=\frac{y}{x^k},\ \zeta=\frac{z}{x^l}.$$

3D wall jet of Prandtl BLEs: exponents

$$x^{2-3k+2l}\int_{-\infty}^{+\infty}d\zeta\int_{0}^{+\infty}f_\eta\left[\zeta f_\eta-\varphi_\eta\right]d\eta=\text{const}\;\Rightarrow\boxed{2-3k+2l=0}$$

In the plane of symmetry:



$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2},$$

$$u\frac{\partial W}{\partial x}+v\frac{\partial W}{\partial y}+W^2=\nu\frac{\partial^2 W}{\partial y^2},$$

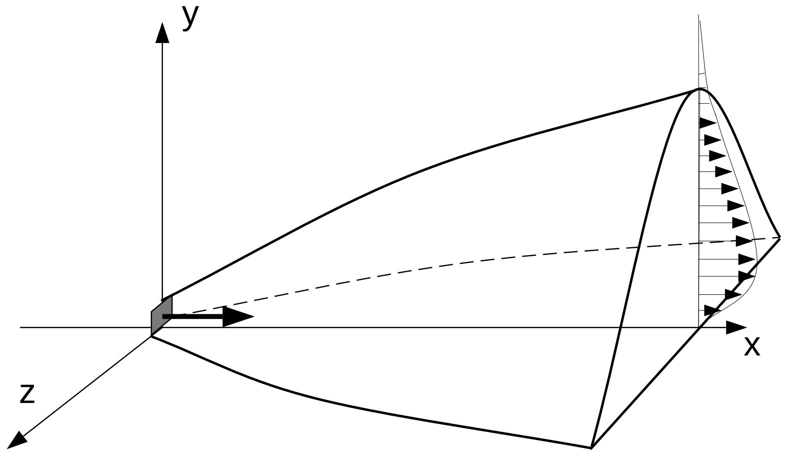
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+W=0,\;W=\frac{\partial w}{\partial z},$$

$$y=0:\;u=v=W=0,$$

$$y=\infty:\;u=W=0.$$

Non-reversed velocity profiles $\Rightarrow k\in\left(\frac{3}{4},\;+\infty\right),\;l\in\left(\frac{1}{8},\;+\infty\right)$

3D wall jet of parabolized NSEs



Parabolized NSEs $(\Delta = \partial_y^2 + \partial_z^2)$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\Delta u,$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta v,$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta w,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$y = 0: \;\; u = v = w = 0, \;\; y = \infty: \;\; u = w = 0,$$

$$z = 0: \;\; \frac{\partial u}{\partial z} = \frac{\partial w}{\partial z} = 0, \;\; z = \pm\infty: \;\; u = w = 0.$$

Self-similarity $(u = \Psi^1_y,\ w = \Psi^2_y,\ v = -\Psi^1_x - \Psi^2_z)$:

$$\Psi^1 = \nu x^{1-k} f(\eta,\zeta)\,, \;\; \Psi^2 = \nu\varphi(\eta,\zeta)\,, \;\; p = \rho\nu^2 x^{-2k} g(\eta,\zeta) \;\; \text{with } \eta = \frac{y}{x^k},\; \zeta = \frac{z}{x^k}$$

$$x^{2-k}\int_{-\infty}^{+\infty} d\zeta \int_0^{+\infty} f_\eta\left[(1-k)\eta f_\eta + (1-k)f - k\zeta f_\zeta + \varphi_\zeta\right] d\eta = \text{const} \;\Rightarrow\; \boxed{k = 2}$$

Wall jets in non-Newtonian fluids

Ostwald - de Waele power law $(n > 1)$:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} = -\frac{\partial p \delta_{ik}}{\partial x_k} + \frac{\partial \sigma'_{ik}}{\partial x_k}, \quad \sigma'_{ik} = \kappa \left| \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right|^{n-1} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

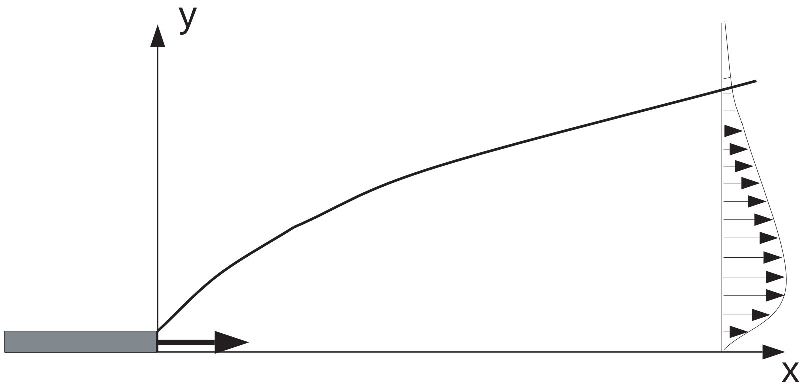
2D Prandtl equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$y = 0: \; u = v = 0,$$

$$y = \infty: \; u = 0.$$



Result: no hidden invariances!

3D Wall jet in non-Newtonian fluids

Governing equations:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}+\frac{\partial}{\partial z}\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z},$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p'}{\partial y}+\frac{\partial}{\partial x}\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y},$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p'}{\partial z}+\frac{\partial}{\partial x}\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Self-similarity:

$$u=x^{\frac{k(1+n)-1}{n-2}}f_u\left(\eta,\zeta\right),\qquad v=x^{\frac{k(2n-1)-(n-1)}{n-2}}f_v\left(\eta,\zeta\right),\ \eta=\frac{y}{x^k},\ \zeta=\frac{z}{x^k}$$

$$w=x^{\frac{k(2n-1)-(n-1)}{n-2}}f_w\left(\eta,\zeta\right),\qquad p'=x^{2\frac{k(2n-1)-(n-1)}{n-2}}f_p\left(\eta,\zeta\right).$$

Hidden invariant:

$$x^{2k+\frac{3kn-2}{n-2}}\int_{-\infty}^{+\infty}d\zeta\int_0^{+\infty}\left[\eta f_u^2-f_uf_v+\left|\frac{\partial f_u}{\partial \eta}\right|^{n-1}\frac{\partial f_u}{\partial \eta}\right]d\eta=\text{const}\ \Rightarrow\ \boxed{k=\frac{2}{5n-4}}$$

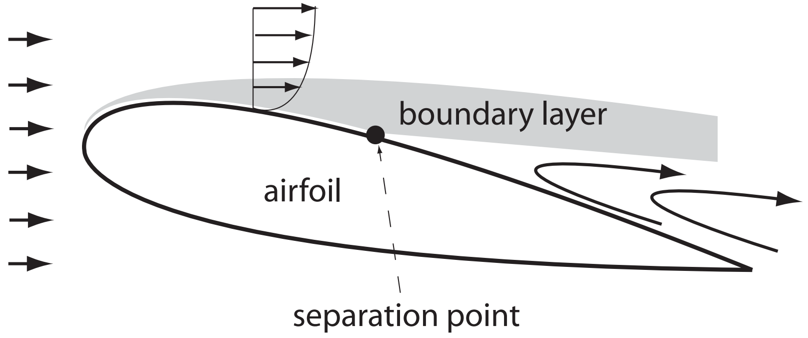
Conclusions: self-similarities in wall jets

- A systematic approach of constructing self-similar solutions of the second kind is demonstrated on the example of wall jets
- The classical result of Akatnov (1953) and Glauert (1956) is reproduced, and uniqueness of the self-similarity exponent is demonstrated
- This new approach allowed to determine the hidden invariances for 3D wall jets in Newtonian and non-Newtonian fluids

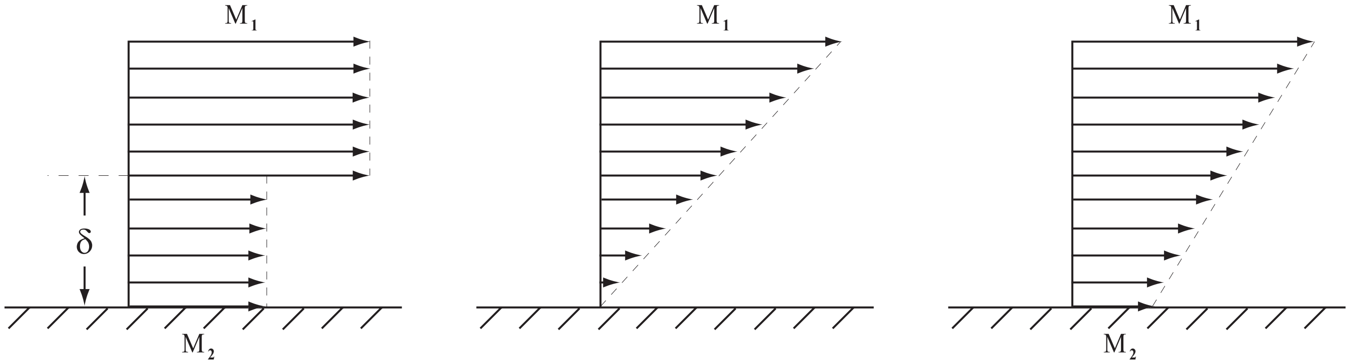
On upstream influence in supersonic flows

Upstream influence:

- Experimental observation: Ferri (1939)
- Theoretical conjecture: Howarth (1948)

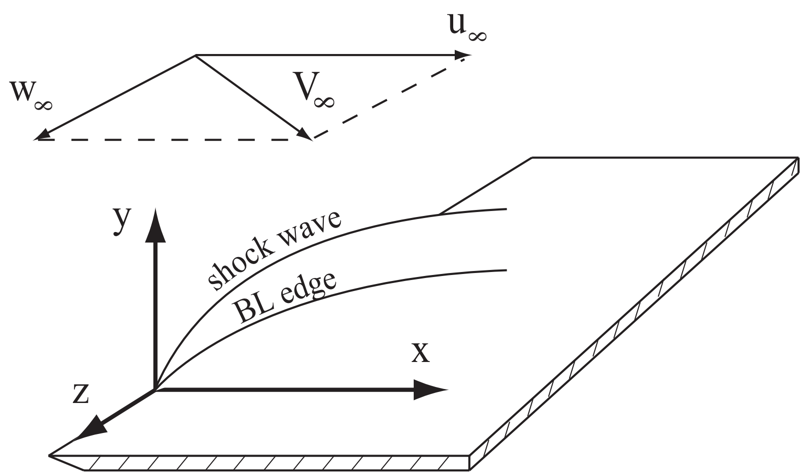


Previous approximate models:



(a) Tsien (1949) (b) Lighthill (1950) (c) Lighthill (1953)

Problem formulation: 3D boundary layer



$$L\rho = -\rho \, \mathrm{div}\mathbf{v},$$

$$\rho L u = -\Delta^2 \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y}\right],$$

$$\rho L w = -\Delta^2 \frac{\partial p}{\partial z} + \frac{\partial}{\partial y}\left[\mu \frac{\partial w}{\partial y}\right],$$

$$\rho L H = 2\;\Delta^2 \frac{\partial p}{\partial t} + \frac{\partial}{\partial y}\left[\mu \frac{\partial H}{\partial y}\right],$$

where $L = \partial_t + \mathbf{u}\cdot\nabla$, $\Delta^2 = T_r/\gamma M_\infty^2$.

Boundary conditions:

$$y = 0 : u = 0, v = v_w, w = 0, H = H_w$$

$$y = \infty : u = 1, w = w_e, H = H_e$$

Interaction condition:

$$\frac{1}{\sqrt{1+w_e^2}}\left(\frac{\partial \delta^*}{\partial t} + \frac{\partial \delta^*}{\partial x} + w_e \frac{\partial \delta^*}{\partial z}\right) = f(p) = \frac{pp_r - 1}{M_\infty}\left[\gamma^2 + \frac{\gamma(\gamma+1)}{2}(pp_r - 1)\right]^{-1/2}$$

“Incompressible form”

Transformation $(t, x, y, z) \to \left(\tau = t, \xi = x, \eta = \int_0^y \rho \, \mathrm{d}\widehat{y}\right)$:

$$0 = \frac{\partial u}{\partial \xi} + \frac{\partial \widetilde{v}}{\partial \eta} + \frac{\partial w}{\partial \zeta},$$

$$Lu = -\frac{\Delta^2}{\rho} \frac{\partial p}{\partial \xi} + \frac{\partial}{\partial \eta} \left[\rho\mu \frac{\partial u}{\partial \eta}\right],$$

$$Lw = -\frac{\Delta^2}{\rho} \frac{\partial p}{\partial \zeta} + \frac{\partial}{\partial \eta} \left[\rho\mu \frac{\partial w}{\partial \eta}\right],$$

$$LH = 2\ \Delta^2 \frac{\partial p}{\partial \tau} + \frac{\partial}{\partial \eta} \left[\rho\mu \frac{\partial H}{\partial \eta}\right],$$

where $\widetilde{v} = \eta_t + u\eta_x + w\eta_z + \rho v$. Thus one can introduce stream-functions

$$u = \psi_\eta,\ \widetilde{v} = -\psi_\xi - \phi_\zeta,\ w = \phi_\eta$$

Boundary layer thickness:

$$\delta^* = \frac{\gamma - 1}{2\gamma\Delta^2} \frac{I}{p}, \quad I \equiv \int_0^{+\infty} \left[H - u^2 - w^2\right] \mathrm{d}\eta$$

Characteristic analysis

Transformation

$$(\tau, \xi, \eta, \zeta) \rightarrow (\Omega(\tau, \xi, \zeta), \eta)$$

yields a system for $\delta^*_\Omega, p_\Omega, \psi_\Omega, \phi_\Omega, H_\Omega$, which after solving for p_Ω produces:

$$N \frac{\mathrm{d}p}{\mathrm{d}\Omega} = \gamma p \left\{ \frac{\sqrt{1+w_e^2}}{\widetilde{\Delta}_e} p f(p) - \widetilde{f} \right\}, \quad \widetilde{\Delta}_e = \left. \Omega_\tau + \psi_\eta \Omega_\xi + \phi_\eta \Omega_\zeta \right|_{\eta=+\infty}$$

The characteristic equation:

$$\boxed{N = \frac{\gamma - 1}{2} \int_0^{+\infty} \frac{(H - u^2 - w^2)^2}{[a - u\cos(\omega) - w\sin(\omega)]^2} \, \mathrm{d}\eta - \int_0^{+\infty} (H - u^2 - w^2) \, \mathrm{d}\eta = 0}$$

a is the speed and ω is the direction angle of disturbances propagation:

$$a = \Omega_\tau \left[\Omega_\xi^2 + \Omega_\zeta^2\right]^{-1/2}, \quad a\cos\omega = -\frac{\Omega_\tau \Omega_\xi}{\Omega_\xi^2 + \Omega_\zeta^2}, \quad a\sin\omega = -\frac{\Omega_\tau \Omega_\zeta}{\Omega_\xi^2 + \Omega_\zeta^2}.$$

Application: a hypersonic yawed wing[a]

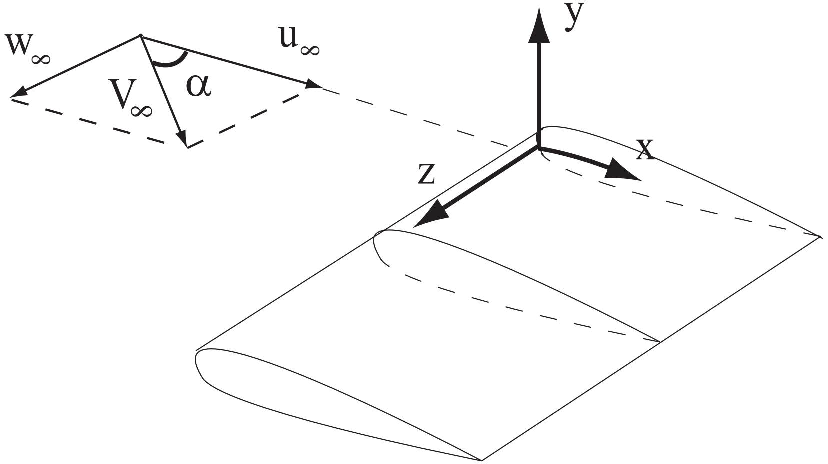


Figure 1: Yawn wing

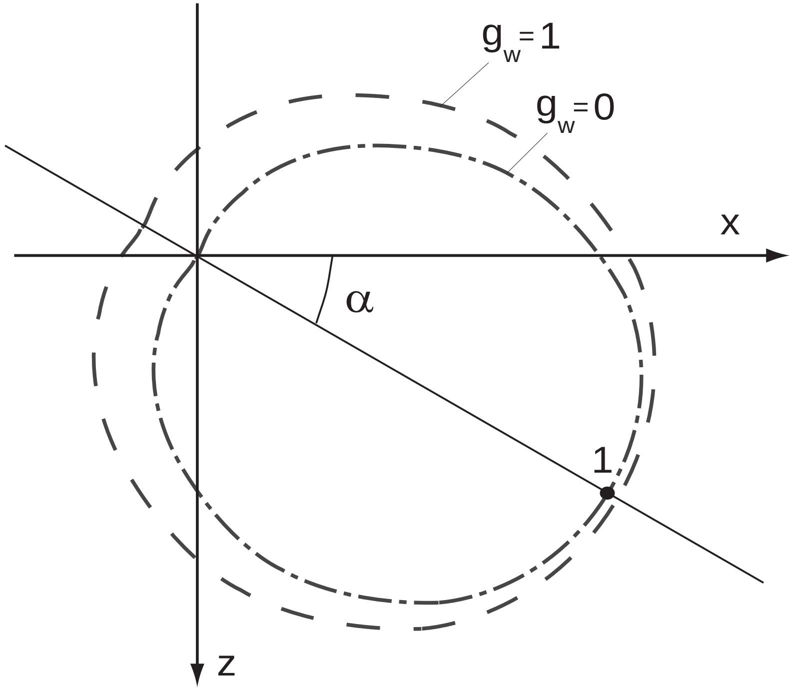


Figure 2: Directional diagram

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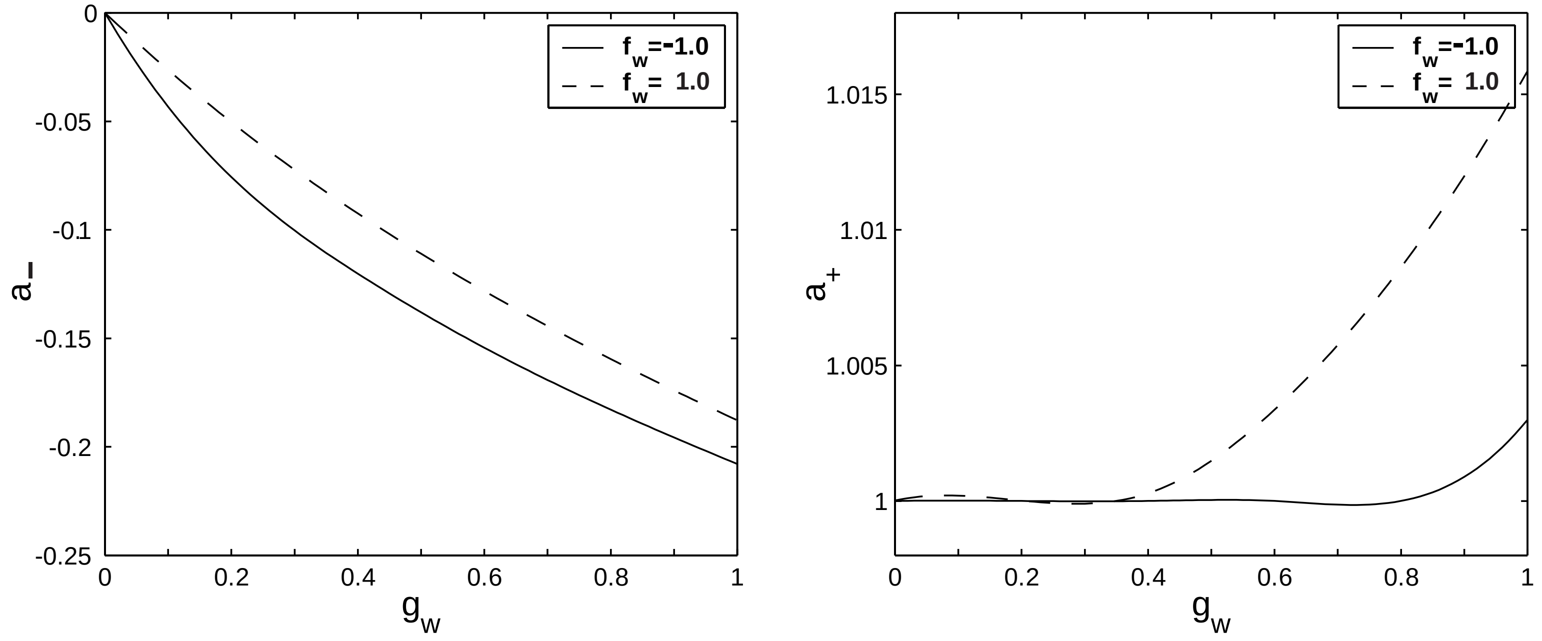
[a] Control parameters: temperature, $g_w \in [0, 1]$, and mass flux, $f_w \in [-1, 1]$

Influence of temperature, g_w, and mass flux, f_w



(a) Upstream speed a_-.

(b) Downstream speed a_+.

Conclusions: upstream influence

- We developed a rigorous (in asymptotic sense) description of upstream influence using theory of characteristics
- Compared to previous *ad hoc* endeavors the theory is now applicable both to linear and nonlinear regimes of disturbances evolution
- The approach allowed a quantitative analysis of the influence of active flow control