Dissipation-Induced Instability Phenomena

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Outline:

- What is a dissipation-induced instability?
  - Theory in finite dimensions
  - Geometric picture
  - General definitions
- Passage to infinite-dimensional systems
  - Various complications
- Infinite-dimensional example: baroclinic instability
  - Set-up and main theorem
  - Hamiltonian structure
  - Linear stability analysis
  - Nonlinear stability analysis
  - Existence theory

General question:

On the abstract level (say, in the Hamiltonian setting) the subject of this talk is the study of the effect of non-conservative perturbations on the dynamics determined by the Hamiltonian, a map $H : T^*Q \to \mathbb{R}$ from the phase space of a mechanical system, namely, a cotangent bundle T^*Q, to a linear space, the real numbers $\mathbb{R}$ (or the energy-momentum map $H \times \mathbf{J} : T^*Q \to \mathbb{R} \times \mathbb{R}^m$).

The non-conservative nature of perturbations is understood as resulting in dynamics whose time evolution satisfies $\mathrm{d}_t H \neq 0$ in general.

The most expensive example:

Figure 1: Explorer I.

Other physical examples

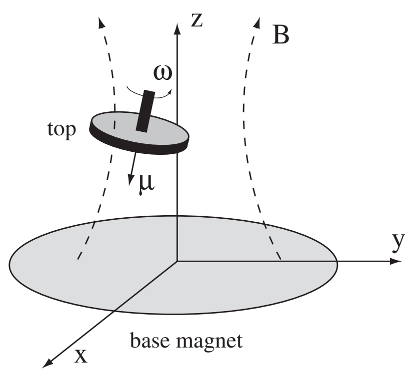


Figure 2: Levitron

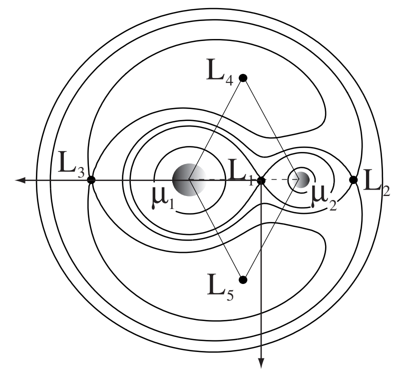


Figure 3: 3-body problem

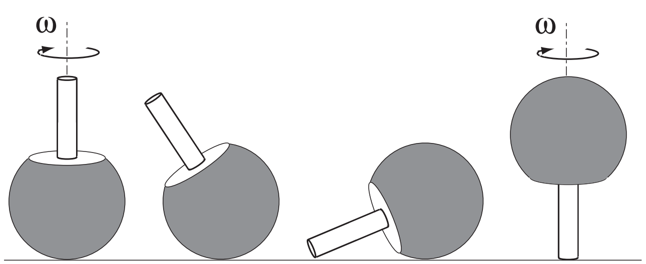


Figure 4: Tippe-top

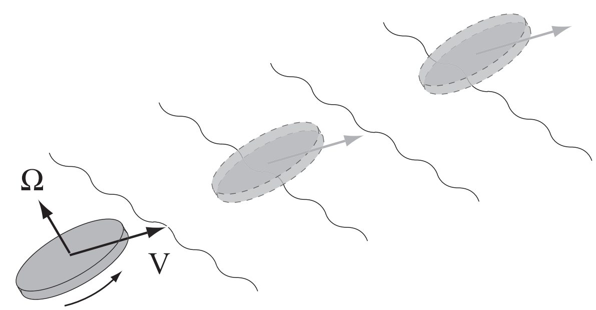


Figure 5: Skipping stone

What is a dissipation-induced instability?

Theory in finite dimensions

Theorem 1. ***(Thomson and Tait 1879, Chetayev 1961)****: if a system with an unstable potential energy is stabilized with gyroscopic forces, then this stability is lost after the addition of arbitrarily small dissipation.*

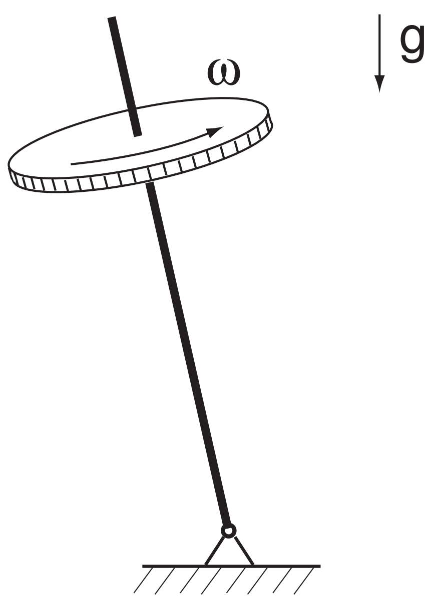


Figure 6: Lagrange top.

Dynamics is governed by

$$\ddot{z}_1 + g\dot{z}_2 - d\dot{z}_1 + c_1 z_1 = 0,$$
$$\ddot{z}_2 - g\dot{z}_1 - d\dot{z}_2 + c_2 z_2 = 0,$$

where $\mathbf{z} = (z_1, z_2)$ is a perturbation, which has an unstable equilibrium at the origin if $g = 0$ and $c_i < 0$, but can be stabilized by the gyroscopic forces if $|g| > \sqrt{-c_1} + \sqrt{-c_2}$.

Theorem 2. ***(Merkin 1956)****: the introduction of non-conservative linear forces into a system with a stable potential and with equal frequencies destroys the stability regardless of the form of nonlinear terms.*

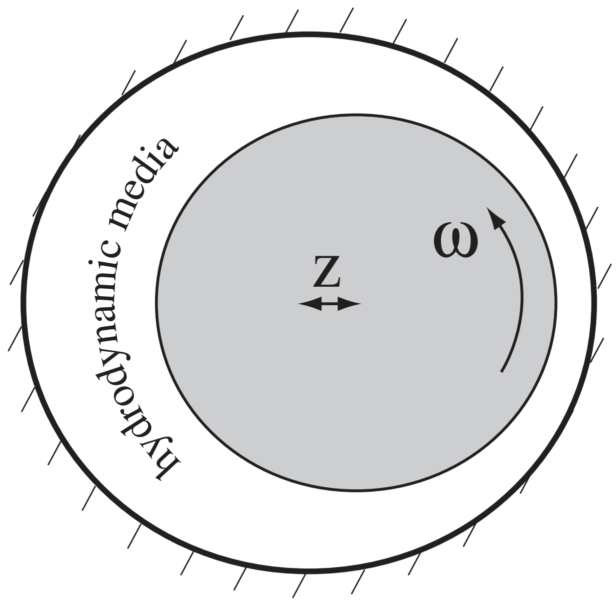


Figure 7: Rotating shaft (Kapitsa 1939).

Dynamics of a perturbation $\mathbf{z}$ is governed by

$$\ddot{z}_1 + p\, z_2 + c\, z_1 = 0,$$
$$\ddot{z}_2 - p\, z_1 + c\, z_2 = 0.$$

The addition of non-zero *non-conservative positional forces* (that is, $p \neq 0$) to a system with a stable potential energy makes it unstable.

General definitions

Definition 1. *A set of non-conservative forces acting on a mechanical system with a relative equilibrium is called* ***dissipative*** *if under the action of these forces and in the absence of the forces which work against these non-conservative forces in order to maintain the (relative) equilibrium, the total mechanical energy of the whole physical system decreases.*

Definition 2. *A conservative system with a Lyapunov stable (relative) equilibrium is said to suffer from* ***dissipation-induced instability***[a] *if the introduction of dissipative forces destabilizes this equilibrium in the Lyapunov sense.*

[a] The term is introduced by Bloch, Krishnaprasad, Marsden, and Ratiu (1994)

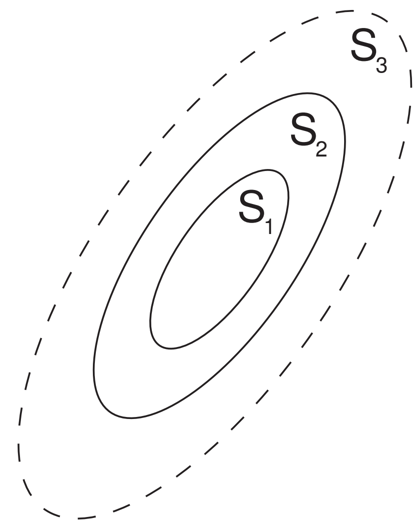
“Thermodynamics”: the total energy, $U = H + \mathcal{Q}$

H is a mechanical, $\mathcal{Q}$ is a non-mechanical (e.g. thermal) energy

Lagrange top:

$S_2 : \mathrm{d}U_2 = 0,\ \mathrm{d}\mathcal{Q}_2 > 0,\ \mathrm{d}H_2^{\text{total}} < 0$

with $\mathrm{d}H_2^{\text{perturb.}} < 0,\ \mathrm{d}H_2^{\text{b. state}} < 0.$



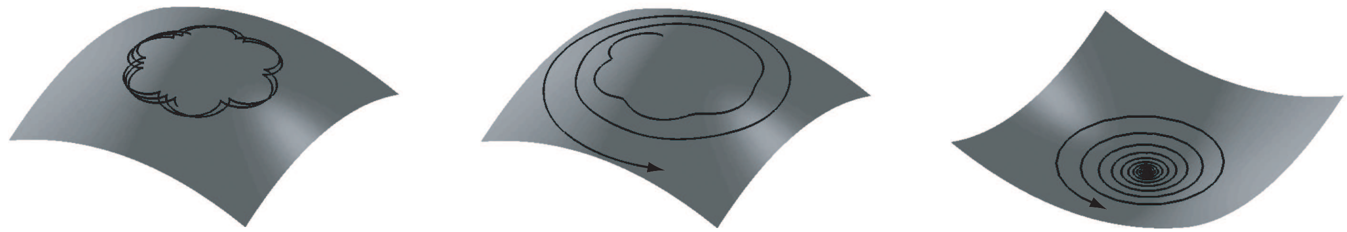
Rotating shaft:

$S_2 : \mathrm{d}U_2 = 0,\ \mathrm{d}\mathcal{Q}_2 > 0,\ \mathrm{d}H_2^{\text{total}} < 0$

with $\mathrm{d}H_2^{\text{perturb.}} > 0,\ \mathrm{d}H_2^{\text{b. state}} < 0.$

The above two key cases – Lagrange top and rotating shaft – evidently span all the possibilities based on the above energy considerations.

Geometric picture: second variation of $H = \frac{1}{2}\dot{\mathbf{z}}^T\dot{\mathbf{z}} + \frac{1}{2}\mathbf{z}C\mathbf{z}$



(a) Gyroscopically stabilized system with an unstable potential.

(b) Destabilization of gyroscopically stable system.

(c) Destabilizing effect of positional forces.

Figure 8: Projection of dynamics onto potential energy surface.

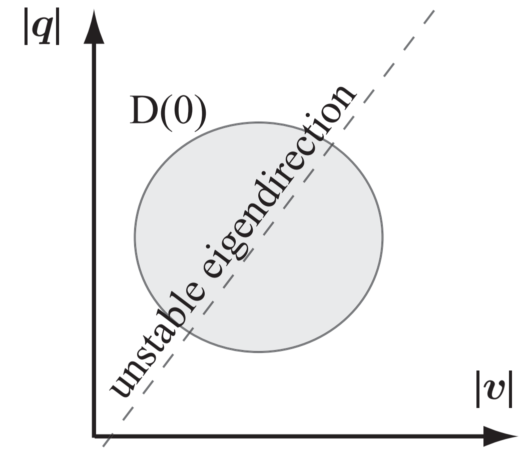
unstable potential	$\rightarrow$ indefinite $\delta^2 H$	$\rightarrow$ dissipative forces
stable potential	$\rightarrow$ definite $\delta^2 H$	$\rightarrow$ positional forces

Table 1: Geometric precursor.

Picture in phase space: $\mathbf{z} = (q_1, v_1, q_2, v_2)^T$

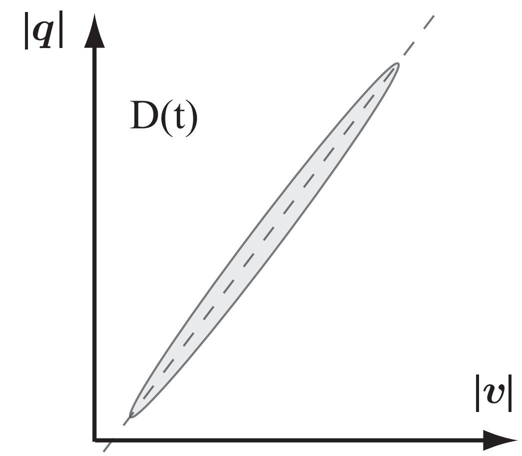
Regular dissipation:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -c_1 & -d & 0 & -g \\ 0 & 0 & 0 & 1 \\ 0 & g & -c_2 & -d \end{pmatrix} \mathbf{z}.$$



Positional forces:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -c & 0 & -p & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & -c & 0 \end{pmatrix} \mathbf{z}.$$



Picture in spectral space

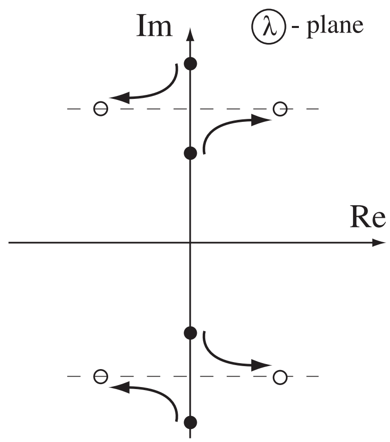


Figure 9: Splitting

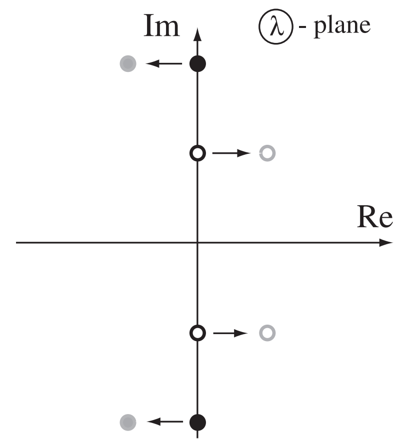


Figure 10: Dissipation effect

Theorem 3. *Consider a Hamiltonian system, which has an equilibrium at* $(\mathbf{0}, \mathbf{0})$*; assume it has a 1:1 resonance. Then, this equilibrium is destabilized (I) by arbitrarily small dissipative forces if the second variation* $\delta^2 H\big|_{(\mathbf{0},\mathbf{0})}$ *is indefinite, or (II) by arbitrarily small positional forces if* $\delta^2 H\big|_{(\mathbf{0},\mathbf{0})}$ *is definite.*

Connection to singularity theory

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} i+\epsilon & 1 \\ \mu + i\nu & i+\epsilon \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

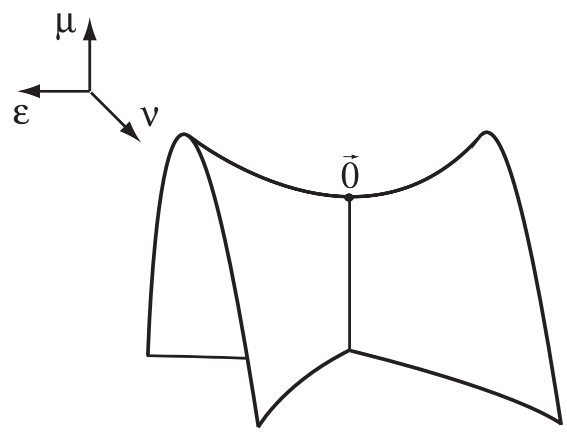


Figure 11: Whitney umbrella.

Conclusion: the codimension-1 Hamiltonian Hopf bifurcation is a singular limit of the codimension-3 dissipative normal form.

Passage to infinite dimensions: complications

- Since the corresponding field equations are usually written not in Lagrangian but rather in Eulerian variables, then one cannot readily identify the type of forces in the infinite-dimensional case.
- Absence of an immediate analog of an infinite-dimensional version of Lyapunov's indirect method (based on the spectrum of linear approximation)
- Establishing the link between existence and nonlinear stability turns out not to be the most natural thing to do since these two analyses are often done in different functional settings.

Infinite-dimensional example

Baroclinic instability: large-scale instability of the baroclinic zonal currents (westerly winds) in mid-latitudes.

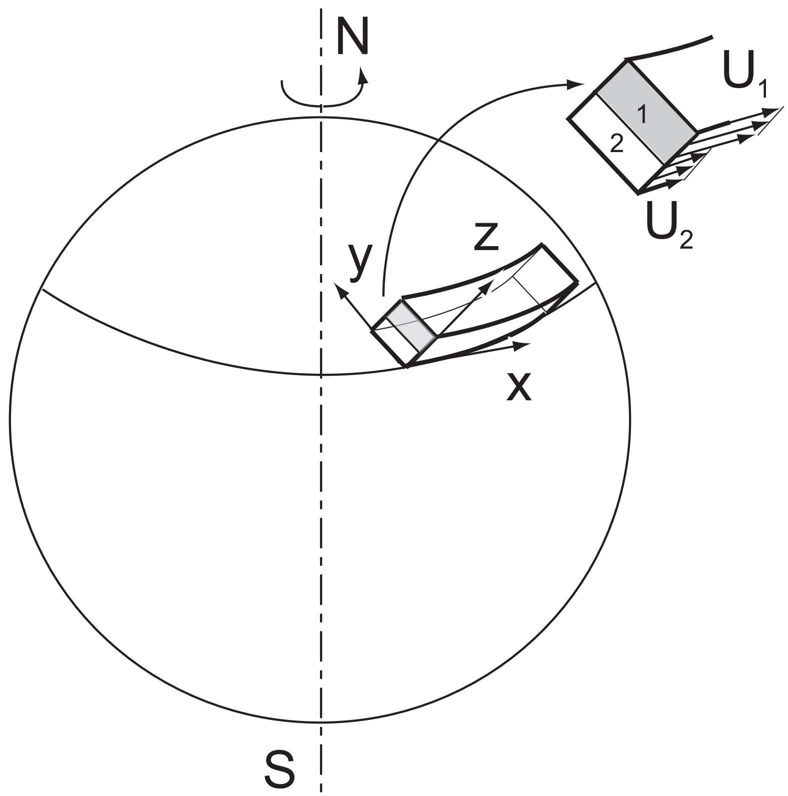


Figure 12: Geometry of the problem (Phillips, 1954).

Differential formulation

Introducing definition of *potential vorticity*,

$$q_i = \nabla^2 \psi_i + (-1)^i F(\psi_1 - \psi_2) + \beta y = \zeta_i + \beta y,$$

where ζ_i is the usual vorticity, $\nabla = \mathbf{i}\partial_x + \mathbf{j}\partial_y$ is a two-dimensional gradient ($\nabla^2 = \Delta$) and the stream-functions ψ_i in the i-th layer are related to velocities by $\mathbf{v}_i = \mathbf{e}_z \times \nabla \psi_i = (-\partial_y \psi_i, \partial_x \psi_i)$, the problem is posed in the form

$$\partial_t q_i + \mathbf{v} \cdot \nabla q_i = -r \Delta \psi_i, \ (x, y) \in \mathbf{D},$$

$$y = 0, 1: \ \partial_x \psi_i = 0 \ \& \ \frac{\partial}{\partial t} \int_{-1}^{1} u_i \mathrm{d}x = 0.$$

Main theorem

Theorem 4. *In the class of classical solutions, the equilibrium* $\psi_i^e = -U_i y$ *of the Hamiltonian quasi-geostrophic two-layer* β*-plane system without dissipation,* $r = 0$*, experiences a dissipation-induced instability in the parameter range* $\left[2\left(1+\sqrt{2}\right)\right]^{-1/2} < |U_1 - U_2|F/2\beta < 1/2$ *in a sense of Definition 2 when an arbitrarily small dissipation effect,* $r > 0$*, is added.*

Physical interpretation: if one is predicting the appearance of a baroclinic instability by measuring the velocity difference $U_c = |U_1 - U_2|$ based on the Hamiltonian formulation, the error of predicting the critical bifurcation parameter will be around 10%.

Linear stability analysis (Romea 1977), $\psi_i = -U_i y + \phi_i$

$$(\partial_t + U_j \partial_x)\left[\nabla^2 \phi_j + (-1)^j F(\phi_1 - \phi_2)\right] + \partial_x \phi_j \left[\beta - (-1)^j F(U_1 - U_2)\right] + r\nabla^2 \phi_j = 0.$$

- Hamiltonian case ($r \equiv 0$):

$$\frac{U_c F}{2\beta} = \frac{1}{\frac{a^2}{F}\left(4 - \frac{a^4}{F^2}\right)^{1/2}}$$

ex-m at $\frac{a^2}{F} = \sqrt{2}$: $\boxed{\frac{U_c F}{2\beta} = \frac{1}{2}}$

- Dissipative case ($r \to 0$):

$$\frac{U_c F}{2\beta} = \frac{1}{\frac{a}{\sqrt{F}}\left(1 + \frac{a^2}{F}\right)\left(2 - \frac{a^2}{F}\right)^{1/2}}$$

at $\frac{a^2}{F} = \sqrt{2}$: $\boxed{\frac{U_c F}{2\beta} = \frac{1}{\sqrt{2}\sqrt{1+\sqrt{2}}}}$

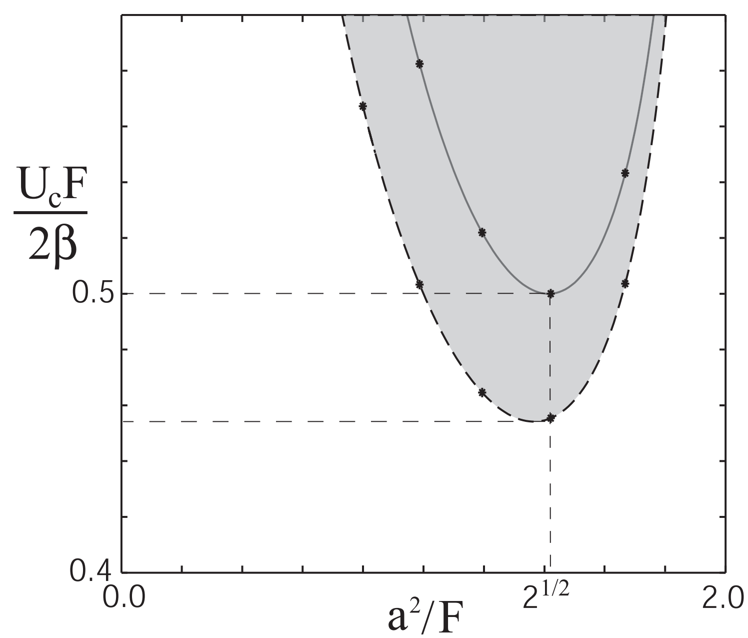


Figure 13: Hamiltonian case (solid line); dissipation case (dashed line).

Hamiltonian structure: $\psi_i = \psi_i^e + \phi_i$

$$H = \frac{1}{2} \int\limits_{\mathbf{D}} \left[F\left(\psi_1 - \psi_2\right)^2 + \left(\nabla\psi_i\right)^2 \right] \mathrm{d}\mathbf{x}, \;\; \dot{H} = r \int\limits_{\mathbf{D}} \left\{ \psi_i^e \nabla^2 \phi_i - \left(\nabla\phi_i\right)^2 \right\} \, \mathrm{d}\mathbf{x}.$$

In a symplectic form:

$$\frac{\partial q_i}{\partial t} = J \frac{\delta \widehat{H}}{\delta q_i}, \quad J = -\partial(q_i, \cdot),$$

with restricted Hamiltonian

$$\widehat{H} = H + \lambda_1^i \int_{-1}^{+1} u_i|_{y=0} \, \mathrm{d}x + \lambda_2^i \int_{-1}^{+1} u_i|_{y=1} \, \mathrm{d}x,$$

where $\lambda_1^i = -\left.\psi_i\right|_{y=0}$ and $\lambda_2^i = -\left.\psi_i\right|_{y=1}$.

Casimirs – the null eigenvectors of J:

$$J \frac{\delta \mathcal{C}}{\delta q_i} = 0 \;\Rightarrow\; \partial\left(q_i, \frac{\delta \mathcal{C}}{\delta q_i}\right) = 0 \;\Rightarrow\; \frac{\delta \mathcal{C}}{\delta q_i} = C'(q_i) \;\Rightarrow\; \boxed{\mathcal{C} = \iint_{\mathbf{D}} C(q_i) \mathrm{d}x \mathrm{d}y}$$

Nonlinear stability analysis: Hamiltonian case

Theorem 5. *If the convexity estimates*

$$Q_1(\delta q) \leq H(q^e + \delta q) - H(q^e) - \delta H(q^e),$$
$$Q_2(\delta q) \leq C(q^e + \delta q) - C(q^e) - \delta C(q^e),$$

hold, which define the norm $\| \cdot \|$*, and if* H_c *is continuous in this norm, then the stability estimate*

$$\|\delta \mathbf{q}(t)\|^2 = Q_1(\delta \mathbf{q}) + Q(\delta \mathbf{q}) \leq (C_1 + C_2)\, \|\delta \mathbf{q}(0)\|^2,$$

holds for all non-zero $\delta \mathbf{q}$*; that is, the equilibrium solution* $\mathbf{q}^e$ *is Lyapunov stable.*

The particular basic state $\psi_i = -U_i y$ is Lyapunov stable if

$$\infty > \psi_i'(q^e) > 0 \;\Rightarrow\; -\tfrac{1}{2} < \tfrac{U_c F}{2\beta} < \tfrac{1}{2}.$$

Nonlinear stability analysis: dissipative case

$$\frac{\partial \boldsymbol{\phi}}{\partial t} = \mathcal{A}\boldsymbol{\phi} + \mathcal{N}(\boldsymbol{\phi}, \boldsymbol{\phi}) \;\Rightarrow\; \boldsymbol{\phi}(t) = \boldsymbol{\phi}(0)e^{\mathcal{A}t} + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{N}(\boldsymbol{\phi}(\tau))\,\mathrm{d}\tau.$$

Properties of the linear operator:

Lemma 1. *The operator $\mathcal{A}$ is elliptic and in $W^{s,p}$ is the infinitesimal generator of an analytic semigroup.*

Properties of the nonlinear operator:

Lemma 2. *The bilinear form $\mathcal{B}(t) = \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{N}(\boldsymbol{\phi}(\tau))\,\mathrm{d}\tau$ in the Duhamel formula grows at most like the square of the norm of $\boldsymbol{\phi}$: if $\|\boldsymbol{\phi}(t)\|_{W^{s,q}} \le \delta e^{\lambda t}$ for all $t \le T$, then there exists a constant C such that*

$$\|\mathcal{B}(t)\|_{W^{s,q}} \le C\left[\delta e^{\lambda t}\right]^2,\; t \le T.$$

Nonlinear stability analysis: dissipative case

Lemma 3. *Suppose* ϕ^e *is an equilibrium solution and a local existence theory*[a] *applies to solutions in* $C^{k,\lambda}$, $0 < \lambda < 1$, *is some neighborhood of* ϕ^e. *Then, if* ϕ^e *is Lyapunov unstable in* $W^{s,p}$ *with* $s \leq k$, *then it is strongly Lyapunov unstable in* $C^{k,\lambda}$.

Theorem 6. *Let* $s \geq 2/q$ *with* $2 < q < \infty$. *If*[b] $U_c F/2\beta > 2^{-1/2}(1+\sqrt{2})^{-1/2}$, *then the equilibrium solution of the quasi-geostrophic system is nonlinearly unstable in* $W^{s,q}$.

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[a] And thus the existence theory allows solutions with an *a priori* bound in $C^{k,\lambda}$ to be continued in time.

[b] This choice of the bifurcation parameter in the quasi-geostrophic system implies that the spectrum of the linear operator $\mathcal{A}$ over Sobolev space contains points in the right half of the complex plane.

Existence of strong solutions

Problem decomposition: the elliptic boundary value problem:

$$\nabla^2\widetilde{\psi}_1 - F\left(\widetilde{\psi}_1 - \widetilde{\psi}_2\right) = \zeta_1, \tag{2a}$$

$$\nabla^2\widetilde{\psi}_2 + F\left(\widetilde{\psi}_1 - \psi_2\right) = \zeta_2, \tag{2b}$$

and the initial value problem for vorticity $\widetilde{\zeta}_i$:

$$\left(\partial_t + \widetilde{u}_i\partial_x + \widetilde{v}_i\partial_y\right)\widetilde{\zeta}_i = -\beta\widetilde{v}_i,\ i = 1, 2, \tag{3a}$$

$$\widetilde{\zeta}_i\Big|_{t=0} = \widetilde{\zeta}_{i0}. \tag{3b}$$

Note the placement of tildas: first one solves the elliptic problem (2) and gets $\widetilde{\boldsymbol{\psi}}$ for given $\boldsymbol{\zeta}$ and then from (3) one determines $\widetilde{\boldsymbol{\zeta}}$. This defines a map

$$\mathcal{T} : \boldsymbol{\zeta} \mapsto \widetilde{\boldsymbol{\zeta}}, \tag{4}$$

Existence of strong solutions[a]

Theorem 7. *(**Local existence in the dissipative case**) Suppose that the initial vorticity* $\boldsymbol{\zeta}^I \in \mathcal{C}^1$ *and* $T^* = L_1^{-1} \ln(1+\epsilon)\,[K_2(1+\epsilon)]^{-1}$. *Then there exists a unique solution to the quasi-geostrophic equations for the time interval* $[0, T^*]$ *with* $\boldsymbol{\zeta} \in \widehat{\mathcal{C}}^1$ and $\boldsymbol{\psi} \in \widehat{\mathcal{C}}_0^{2,\lambda}$.

Theorem 8. *(**Global existence in the Hamiltonian case**) If initial data* $\|\boldsymbol{\zeta}^I\| \in \mathcal{H}^s$, $s \geq 3$, *then there exists a classical solution* $\boldsymbol{\zeta} \in \widehat{\mathcal{C}}_0^1$ *to the quasi-geostrophic equations on any finite time interval,* $0 \leq t \leq T < \infty$.

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[a] By $\mathcal{C}^{s,\lambda}$ we denote the Hölder space of functions bounded and continuous together with their spatial derivatives of order $|\alpha| \leq s \in \mathbf{Z}_+$, the derivatives of order s being uniformly Hölder continuous with exponent $\lambda \in (0,1]$. $\mathcal{H}^s$ is the Sobolev space with generalized derivatives of order $|\alpha| \leq s \in \mathbf{Z}_+$ which are bounded in L^2-norm.

Proof of the main theorem

Proof. Based on Theorems 5-8, the main result – Theorem 4 on dissipation-induced instability in the quasi-geostrophic problem – follows as a corollary, when all these theorems are applied in the class of strong solutions:

- From Theorem 8 we know the existence of a strong solution of the quasi-geostrophic system in the Hamiltonian case, $r = 0$, for all values of all parameters, i.e. everywhere in figure 18.
- From Theorem 5 it follows that the strong solution in the Hamiltonian case is Lyapunov stable for $U_c F/2\beta < 1/2$, i.e. below the solid curve in figure 18.
- From Theorem 6 it follows that the strong solution in the dissipative case is Lyapunov unstable for $2^{-1/2}\left(1+\sqrt{2}\right)^{-1/2} < U_c F/2\beta < 1/2$, i.e. the region between the dashed and solid curves. This result applies regardless of the fact of existence of a strong solution in the dissipative case. The local existence, though, is proved in Theorem 7.

□

Conclusions

- A multitude of physical applications and situations, in which dissipation-induced instabilities occurs, indicates that this particular effect is one of the paramount ones governing instability mechanisms in Nature.

- Clearly, there are many open problems and issues associated with our further understanding both at the fundamental level, e.g. infinite-dimensional systems, and on the applied side, e.g. control.

- Note that we have at our disposal only two categories of systems, Hamiltonian, $\mathrm{d}H = 0$, and dissipative, $\mathrm{d}H < 0$. Therefore, all the instabilities can be divided into two classes: (1) which are accounted for by the Hamiltonian description, and (2) which are due to dissipation.

References

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